

A Double-Minded Fractal

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The fractal in Figure 1 appears to say “Yes” at the topmost level, then “No” at the secondary level, and “Yes” at the tertiary level, and so on, all the way down. We outline how to generate such sets. Variations on these ideas make a fun and instructive project in a linear algebra class. All that is needed is a little familiarity with linear transformations of the plane.

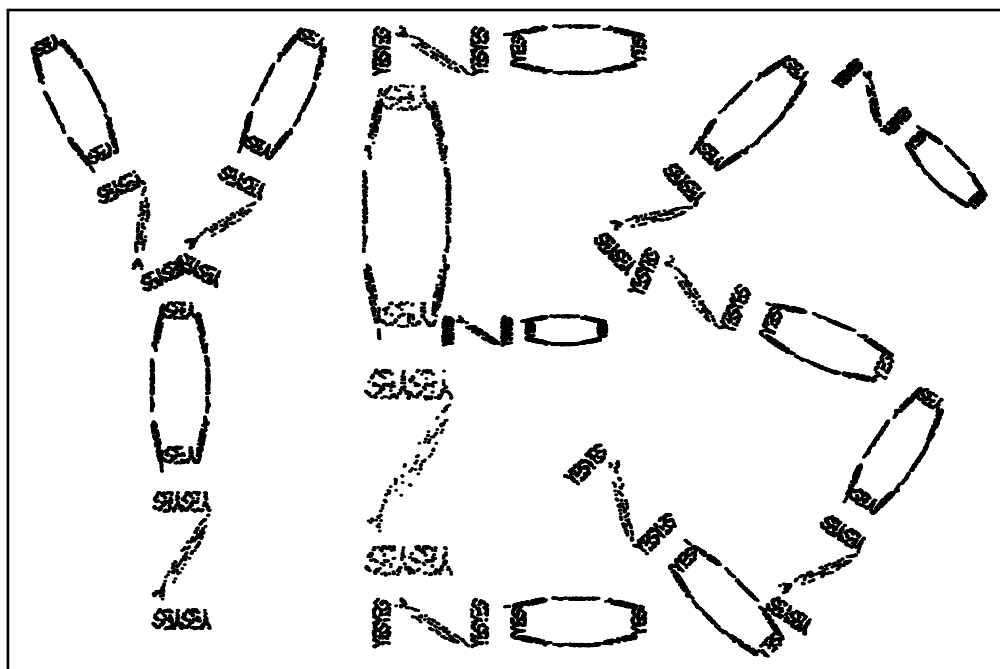


Figure 1: A Yes-No fractal

A Sierpinski prelude

We first consider a simpler fractal example.

Informally, a fractal is a planar set whose parts are similar to its whole. The Sierpinski triangle S of Figure 2(b) is a classic example. At its outermost level, S is contained in an equilateral triangle with vertices $(0, 0)$, $(4, 0)$, and $(2, 2\sqrt{3})$. As indicated in Figure 2(a), S appears to be the union of three smaller equilateral triangles each of side length 2 and each of which is a translate of a $\frac{1}{2}$ -scaled copy of S . We use these observations to sharpen our definition of a fractal. We say that a function T is a contraction of the plane if there is a

2×2 matrix M and a vector b for which $T(x) = Mx + b$ with $|Mx| < |x|$ for all x (except $\mathbf{0}$) in the plane. Let \mathcal{F} be a family of contractions. More formally, we define a fractal A corresponding to \mathcal{F} as a set of points in the plane for which

$$A = \bigcup_{T \in \mathcal{F}} T(A).$$

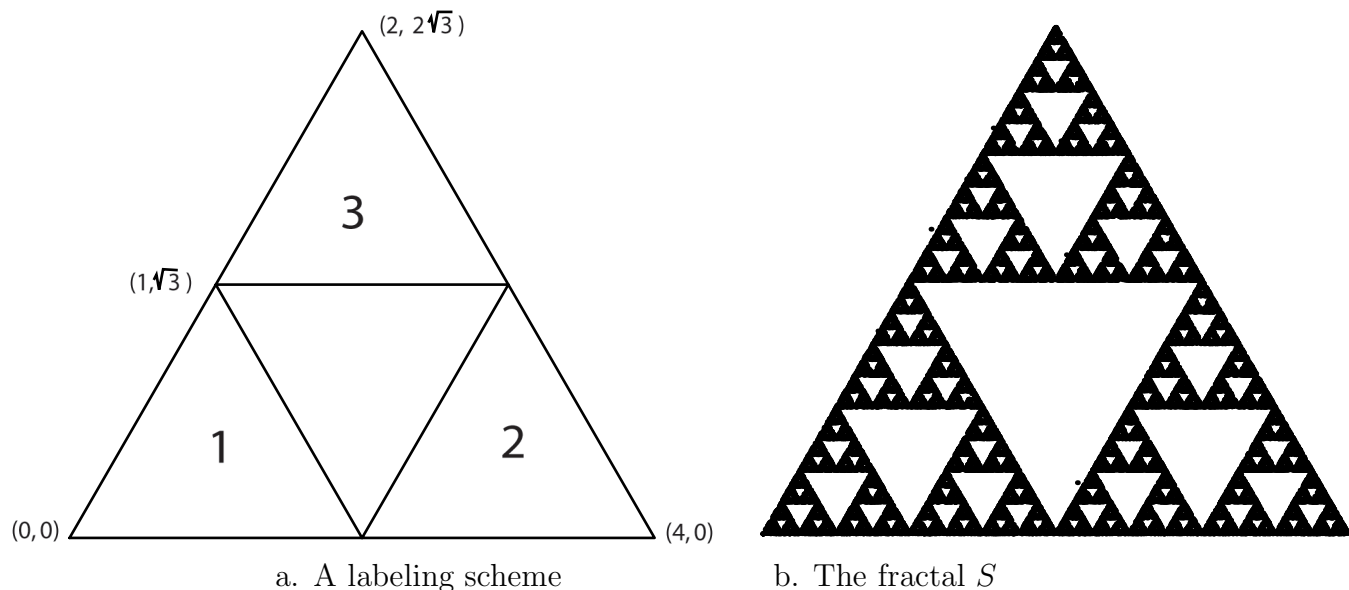


Figure 2: The Sierpinski triangle

A corresponding family of contractions for S has as its members:

$$T_1(x) = x/2, \quad T_2(x) = x/2 + (2, 0), \quad T_3(x) = x/2 + (1, \sqrt{3}).$$

(The matrix M in each of these contractions is $\frac{1}{2}$ times the identity matrix.) According to the labeling scheme of Figure 2(a), T_1 maps S into triangle 1, T_2 maps S into triangle 2, and T_3 maps S into triangle 3. Although we omit a proof, S is indeed equal to $T_1(S) \cup T_2(S) \cup T_3(S)$. One way to graph the fractal corresponding to this family is to start with an arbitrary point X ; for convenience, we take a point inside the large equilateral triangle of Figure 2(a), such as $(0.3, 0.1)$. Randomly choose a contraction T (from the family) and plot $T(X)$, calling the resulting point X . Then repeat the instructions of the last sentence a suitable number of times. As the repetition number increases, so does the resolution of the fractal. A little *Mathematica* code, readily adaptable into any computer algebra system, generates Figure 2(b):

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F[x_, n_] := If[n ≤ 33, x/2,
  If[n ≤ 67, x/2 + {2, 0}, x/2 + {1, √3}]];

Block[{bag, i, X}, X = {0.3, 0.1}; bag = {};
Do[X = F[X, Random[Integer, 100]];
  bag = Append[bag, X], {i, 1, 1000}];
ListPlot[bag, AspectRatio → Automatic]];

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In the above code, the initial value of X is $(0.3, 0.1)$ and the repetition number is 1000 (so as to plot 1000 points). Figure 2(b) is a plot of approximately 30,000 points, all of whose points are very close to points in S , which means that the image is a faithful artistic portrait of the ideal set S .

A Yes-No Construction Outline

To generate Figure 1 we first of all find families of functions corresponding to two fractals, one of which says “Yes” all the way down and the other of which says “No.” On a rectangular grid G whose lower-left and upper-right corners are $(0, 0)$ and $(30, 20)$, we layout a design of the word “Yes,” breaking it into the twelve natural rectangles as indicated in Figure 3. We layout a similar scheme for “No” on G , this time using thirteen natural rectangles to do so.

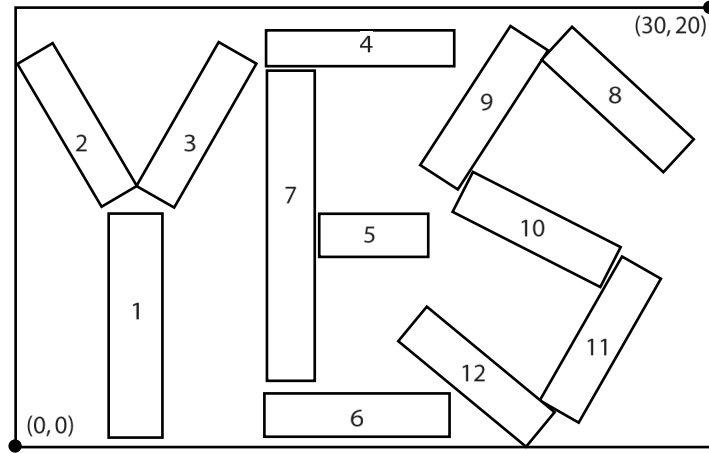


Figure 3: A labeling scheme for “Yes”

Next, for each of these smaller rectangles, find a contraction which transforms G onto the smaller one. For example, the contraction corresponding to rectangle 1 of Figure 3 is

obtained by shrinking G appropriately and then rotating it counterclockwise by 90° and finally translating the result so as to form the stem of the letter “Y”:

$$T_1(x) = \begin{bmatrix} 0 & -0.092 \\ 0.4 & 0 \end{bmatrix} x + \begin{bmatrix} 4.5 \\ 0 \end{bmatrix}.$$

Let \mathcal{Y} be the twelve contractions for “Yes,” and \mathcal{N} be the thirteen contractions for “No.”

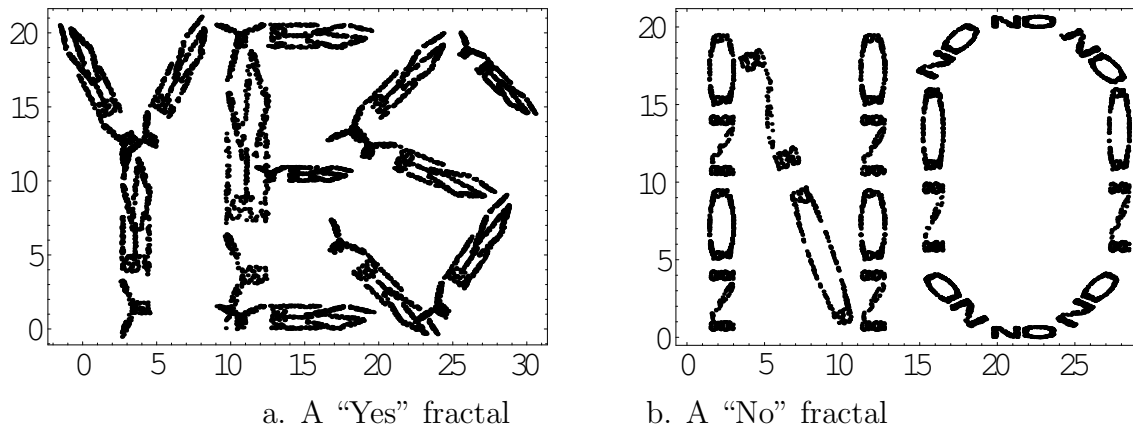


Figure 4: Single-minded fractals

Adaptations of the *Mathematica* code of the previous section generate Figure 4.

With \mathcal{Y} and \mathcal{N} defined, we are ready to generate a double-minded fractal. As before, we start with an arbitrary point X . Randomly choose a contraction T from \mathcal{Y} and randomly choose a contraction U from \mathcal{N} ; plot $T(U(X))$, calling the resulting point X . Repeat the instructions of the last sentence a suitable number of times. The result is Figure 1. That is, we have implicitly defined a family of 156 transformations by composing the 12 and 13 explicitly defined transformations of \mathcal{Y} and \mathcal{N} .

Some related exercises

Finally, here are a few problems to try in the classroom.

- Generate a double-minded fractal that starts with “No” on the outermost level.
- Generate a triple-minded fractal that cycles through the alternatives “Yes-Maybe-No.”
- Adapt the *CAS* code in this note so as to zoom in on any portion of the fractal. For example, Figure 5 zooms in on the left-hand leg of the “N” in “No” in the middle arm of the “E” of “Yes.”

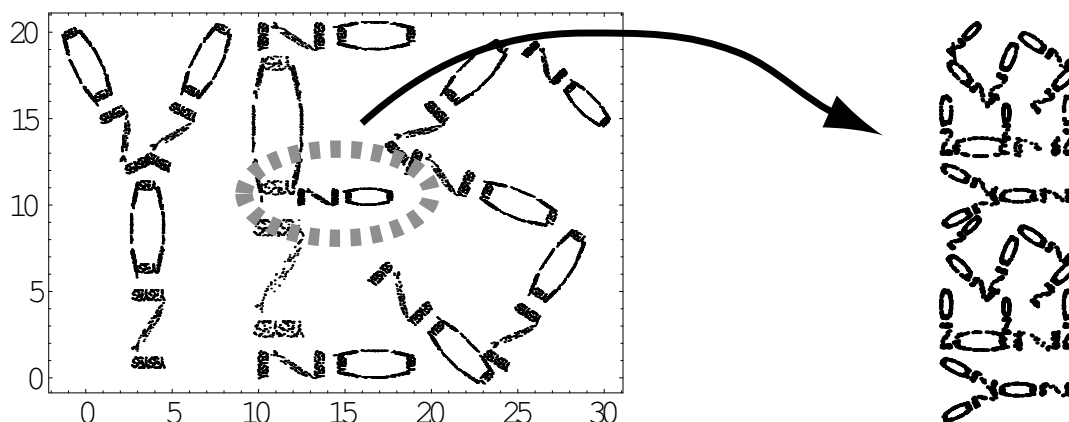


Figure 5: Zooming in on a fractal

Discussion of algorithms to generate fractals, including the randomized method of this note, can be found in any fractal text such as [1] or [2].

References

1. Michael Barnsley, *Fractals Everywhere*, Academic Press, 1988.
2. Gerald A. Edgar, *Measure, Topology, and Fractal Geometry*, Springer-Verlag, 1990.