Computing the Moore-Penrose inverse of a matrix with a Computer Algebra System

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In this presentation *Derive* functions are provided for the computation of the Moore-Penrose inverse of a matrix, as well as for solving systems of linear equations by means of the Moore-Penrose inverse. Making it possible to compute the Moore-Penrose inverse easily with one of the most commonly used Computer Algebra Systems – and to have the blueprint to write such a function in other Computer Algebra Systems or in a matrix programming language such as *Gauss* – may promote the use of generalised inverses in the teaching of linear algebra.

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Definition

For any matrix $\underset{m\times n}{\boldsymbol{A}}$ there exists a unique Moore-Penrose inverse, denoted by $\underset{n\times m}{\boldsymbol{A}^+}$, which satisfies the four conditions

$$\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{A} \tag{1}$$

$$\boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{A}^+ \tag{2}$$

$$\left(\boldsymbol{A}^+\boldsymbol{A}\right)' = \boldsymbol{A}^+\boldsymbol{A} \tag{3}$$

$$\left(\boldsymbol{A}\boldsymbol{A}^+\right)' = \boldsymbol{A}\boldsymbol{A}^+ \tag{4}$$

Some properties

- If $\boldsymbol{A}$ is a nonsingular matrix, we have $\boldsymbol{A}^+ = \boldsymbol{A}^{-1}$.
- $\boldsymbol{A}^+\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{A}^+$ are idempotent matrices
- $\mathrm{r}\left(\boldsymbol{A}\right) = \mathrm{r}\left(\boldsymbol{A}^+\boldsymbol{A}\right) = \mathrm{tr}\left(\boldsymbol{A}^+\boldsymbol{A}\right)$
- If $\mathrm{r}\left(\underset{m\times n}{\boldsymbol{A}}\right) = n$, we have $\boldsymbol{A}^+ = \left(\boldsymbol{A}'\boldsymbol{A}\right)^{-1}\boldsymbol{A}'$ and $\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{I}$
- If $\mathrm{r}\left(\underset{m\times n}{\boldsymbol{A}}\right) = m$, we have $\boldsymbol{A}^+ = \boldsymbol{A}'\left(\boldsymbol{A}\boldsymbol{A}'\right)^{-1}$ and $\boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{I}$

Computation of the Moore-Penrose inverse of a (column) vector

$$\boldsymbol{a}^{+} = \begin{cases} \frac{1}{\boldsymbol{a}'\boldsymbol{a}}\boldsymbol{a}' & \text{if } \boldsymbol{a} \neq \boldsymbol{o} \\ \boldsymbol{o}' & \text{if } \boldsymbol{a} = \boldsymbol{o} \end{cases} \tag{5}$$

A vector is a matrix with only one column and should be declared in *Derive* as such.

```
#1:  MPIV(a) :=
       If DIM(a') = 1
          If (a'·a)↓1↓1 = 0
             0·a'
             a'/(a'·a)↓1↓1
          "This is not a column vector!"
```

If a vector has symbolic elements, the `MPIV` function may possibly not be able to compute its Moore-Penrose inverse.

#1: LOAD(C:\Programme\TI Education\Derive 6\Math\MP.mth)

#2: $a := \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix}$

#3: $\mathrm{MPIV}(a) = \left[\left[\frac{1}{x^2+5}, \frac{2}{x^2+5}, \frac{x}{x^2+5}\right]\right]$

#4: $b := \begin{bmatrix} 2\cdot y \\ 0 \end{bmatrix}$

#5: $\mathrm{MPIV}(b) = \mathrm{IF}\left(y = 0,\ 0\cdot\begin{bmatrix} 2\cdot y \\ 0 \end{bmatrix}',\ \begin{bmatrix} 2\cdot y \\ 0 \end{bmatrix}'\cdot\left(\begin{bmatrix} 2\cdot y \\ 0 \end{bmatrix}'\cdot\begin{bmatrix} 2\cdot y \\ 0 \end{bmatrix}\right)_{1,1}^{-1}\right)$

Computation of the Moore-Penrose inverse of a matrix

`MPI` function starts by calling the `MPIV` function with the first column of $\boldsymbol{A}$. The result is the first row of $\boldsymbol{A}^{+}$ (which is only an intermediate result).

The `MPI` function then proceeds to the second column of $\boldsymbol{A}$ and computes the second intermediate $\boldsymbol{A}^{+}$ by transforming the previous result and appending another row. This is repeated for all columns of $\boldsymbol{A}$. After as many steps as the number of columns of $\boldsymbol{A}$ the `MPI` function has found $\boldsymbol{A}^{+}$.

Note that in each step the `MPIV` function is called. Hence, in the case of symbolic elements the `MPI` function might be unable to compute $\boldsymbol{A}^{+}$.

```
        MPI(A, APLUS, aj, dt, c, bt, J) :=
          Prog
            APLUS := MPIV(A COL [1])
            J := 2
            Loop
              If J > DIM(A')
#2:              RETURN APLUS
              aj := A COL [J]
              dt := aj'·APLUS'·APLUS
              c := (IDENTITY_MATRIX(DIM(A)) - A COL [1, ..., J - 1]·APLUS)·aj
              bt := MPIV(c) + (1 - MPIV(c)·c)/(1 + dt·aj)·dt
              APLUS := APPEND(APLUS - APLUS·aj·bt, bt)
              J :+ 1
```

Application

We consider a system of linear equations (SLE)

$$\underset{m\times n}{\boldsymbol{A}}\ \underset{n\times 1}{\boldsymbol{x}} = \underset{m\times 1}{\boldsymbol{b}}$$

where $\boldsymbol{A}$ is a known coefficient matrix, $\boldsymbol{b}$ a vector of known constants, and $\boldsymbol{x}$ a vector of unknown variables.

A system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent if and only if

$$\boldsymbol{AA}^{+}\boldsymbol{b} = \boldsymbol{b} \tag{7}$$

If a system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent, its general solution is given by

$$\boldsymbol{x} = \boldsymbol{A}^{+}\boldsymbol{b} + \left(\underset{n\times n}{\boldsymbol{I}} - \boldsymbol{A}^{+}\boldsymbol{A}\right)\underset{n\times 1}{\boldsymbol{z}} \tag{8}$$

where $\boldsymbol{z}$ is an arbitrary vector.

The following function `SOLVESLE` (in conjunction with a vector $\boldsymbol{z}$ with elements `z1`, `z2`, …) either solves a system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$, where the matrix $\boldsymbol{A}$ and the vector $\boldsymbol{b}$ have been passed as parameters, or displays a message if the system is inconsistent.

```
#4:   z := VECTOR(VECTOR(APPEND(z, J), i, 1), J, 1, DIM(A'))

      SOLVESLE(A, b) :=
        If A·MPI(A)·b = b
#5:        MPI(A)·b + (IDENTITY_MATRIX(DIMENSION(A')) - MPI(A)·A)·z
           "No solution(s)!"
```

Example I

Consider the SLE $\boldsymbol{Ax} = \boldsymbol{b}$ defined by $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$; $\boldsymbol{b} = \begin{pmatrix} \frac{3}{2} \\ \frac{7}{2} \end{pmatrix}$

$\boldsymbol{A}$ is a nonsingular matrix and therefore $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} \\ \frac{7}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ -\frac{1}{2} \end{pmatrix}$

For any nonsingular matrix $\boldsymbol{A}$ condition (7) is fulfilled since $\boldsymbol{AA}^{+}\boldsymbol{b} = \boldsymbol{AA}^{-1}\boldsymbol{b} = \boldsymbol{b}$.

Moreover, since $\left(\boldsymbol{I} - \boldsymbol{A}^{+}\boldsymbol{A}\right)\boldsymbol{z} = \left(\boldsymbol{I} - \boldsymbol{A}^{-1}\boldsymbol{A}\right)\boldsymbol{z} = \boldsymbol{o}$ for any $\boldsymbol{z}$, the general solution (8) simplifies to $\boldsymbol{x} = \boldsymbol{A}^{+}\boldsymbol{b} = \boldsymbol{A}^{-1}\boldsymbol{b}$.

Example II

Now consider the SLE defined by $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$; $\boldsymbol{b} = \begin{pmatrix} \frac{3}{2} \\ \frac{7}{2} \end{pmatrix}$

$\boldsymbol{A}$ is a singular matrix. Using the Moore-Penrose inverse of $\boldsymbol{A}$ we find that this SLE is inconsistent (note that $\boldsymbol{AA}^{+} \neq \boldsymbol{I}$):

$$\boldsymbol{AA}^{+}\boldsymbol{b} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{25} & \frac{2}{25} \\ \frac{2}{25} & \frac{4}{25} \end{pmatrix} \begin{pmatrix} \frac{3}{2} \\ \frac{7}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} \frac{3}{2} \\ \frac{7}{2} \end{pmatrix} = \begin{pmatrix} \frac{17}{10} \\ \frac{17}{5} \end{pmatrix} \neq \boldsymbol{b}$$

Example III

Finally, consider the SLE defined by $\boldsymbol{A}=\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix};\quad \boldsymbol{b}=\begin{pmatrix}\frac{3}{2}\\ 3\end{pmatrix}$

Using the Moore-Penrose inverse of $\boldsymbol{A}$ we find that this SLE is consistent:

$$\boldsymbol{A}\boldsymbol{A}^{+}\boldsymbol{b}=\begin{pmatrix}\frac{1}{5} & \frac{2}{5}\\ \frac{2}{5} & \frac{4}{5}\end{pmatrix}\begin{pmatrix}\frac{3}{2}\\ 3\end{pmatrix}=\begin{pmatrix}\frac{3}{2}\\ 3\end{pmatrix}=\boldsymbol{b}$$

The general solution now consists of an infinite number of vectors defined by

$$\boldsymbol{x}=\boldsymbol{A}^{+}\boldsymbol{b}+\left(\boldsymbol{I}-\boldsymbol{A}^{+}\boldsymbol{A}\right)\boldsymbol{z}=\begin{pmatrix}\frac{3}{10}\\ \frac{3}{5}\end{pmatrix}+\begin{pmatrix}\frac{4}{5} & -\frac{2}{5}\\ -\frac{2}{5} & \frac{1}{5}\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix}$$

In *Derive* we get the same results by applying the function `SOLVESLE`:

#1: LOAD(C:\Programme\TI Education\Derive 6\Math\MP.mth)

#2: $$\mathrm{SOLVESLE}\left(\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}, \begin{bmatrix}\frac{3}{2}\\ \frac{7}{2}\end{bmatrix}\right)=\begin{bmatrix}\frac{5}{2}\\ -\frac{1}{2}\end{bmatrix}$$

#3: $$\mathrm{SOLVESLE}\left(\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}, \begin{bmatrix}\frac{3}{2}\\ \frac{7}{2}\end{bmatrix}\right)=\text{No solution(s)!}$$

#4: $$\mathrm{SOLVESLE}\left(\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}, \begin{bmatrix}\frac{3}{2}\\ 3\end{bmatrix}\right)=\begin{bmatrix}\frac{4\cdot z1}{5}-\frac{2\cdot z2}{5}+\frac{3}{10}\\ -\frac{2\cdot z1}{5}+\frac{z2}{5}+\frac{3}{5}\end{bmatrix}$$

References

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