

January 2016 Joint Mathematics Meeting
Helping Students See Beyond Calculus
Contributed Paper Session

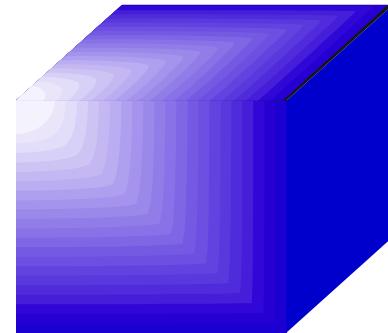
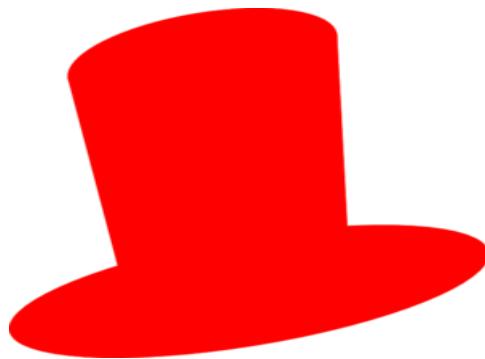
Hats, Hamming and Hypercubes

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Hats, Hamming and Hypercubes



The 3-Hat Game

A team consists of three players.

For one round of the game:

- A **red** or **blue** hat is placed on each player's head.
Hat color is *randomly determined*.

*Each player can see the hats of the other players
but cannot see his or her own hat.*
- No communication of any sort is allowed, except for
the strategy session before the game begins.



All players simultaneously guess the color of their own hats or pass.



Red



Blue

Pass

Win: at least one player guesses *correctly* and no player guesses incorrectly.

Lose: all players pass or at least one player guesses incorrectly.

The 3-Hat Game

We will play 8 rounds of the game.

For example, the hat colors for a player might look like this:

Round 1 **Blue**

Round 2 **Red**

Round 3 **Blue**

Round 4 **Blue**

Round 5 **Red**

Round 6 **Red**

Round 7 **Red**

Round 8 **Blue**

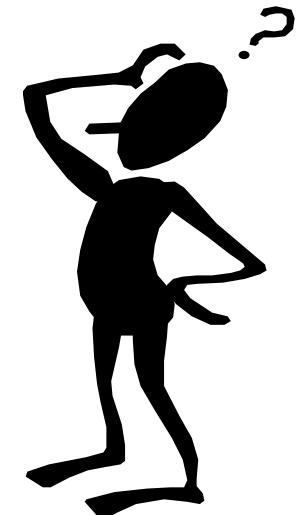
3-Hat Game Strategy?

What strategy should a team use to have
the best chance of winning the game?

Is there strategy we can employ for which
we would expect to win more
than 50% of the time?



With a **best strategy**, how often would the team
expect to win if many rounds of the game are
played?



Eight Possible Hat Combinations

<u>1st</u>	<u>2nd</u>	<u>3rd</u>
-----------------------	-----------------------	-----------------------

B	B	B
---	---	---

B	B	R
---	---	---

B	R	B
---	---	---

B	R	R
---	---	---



R	B	B
---	---	---

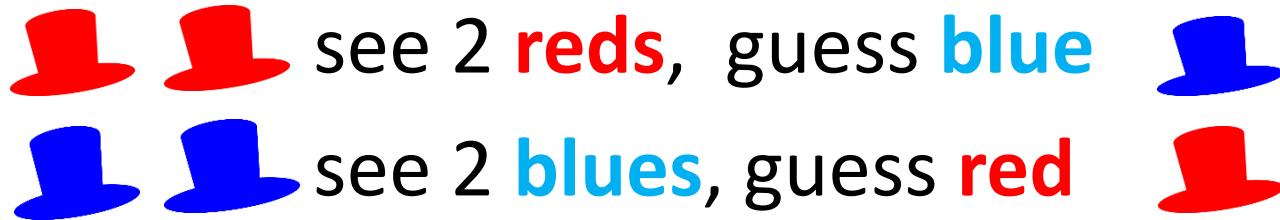
R	B	R
---	---	---

R	R	B
---	---	---

R	R	R
---	---	---

A Best Strategy with 3 Hats

If a player sees **two hats of the same color**,
she guesses the **opposite color** :



If a player sees **one hat of each color**,



Analyzing the Best Strategy

1st

2nd

3rd

B

B

B



Lose

B

B

R



Win

B

R

B



Win

B

R

R



Win

R

B

B



Win

R

B

R



Win

R

R

B



Win

R

R

R



Lose

Binary Representation of Hats

Eight bit strings of length three

1	1	1
1	1	0
1	0	1
1	0	0
0	1	1
0	1	0
0	0	1
0	0	0

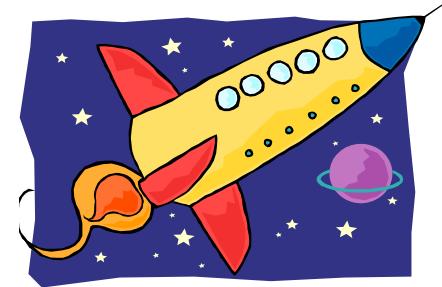
Binary Representation of Hats

and Truth Tables from Symbolic Logic

1	1	1	T	T	T
1	1	0	T	T	F
1	0	1	T	F	T
1	0	0	T	F	F
→			F	T	T
0	1	1	F	T	F
0	1	0	F	T	F
0	0	1	F	F	T
0	0	0	F	F	F

Error-Correcting Codes

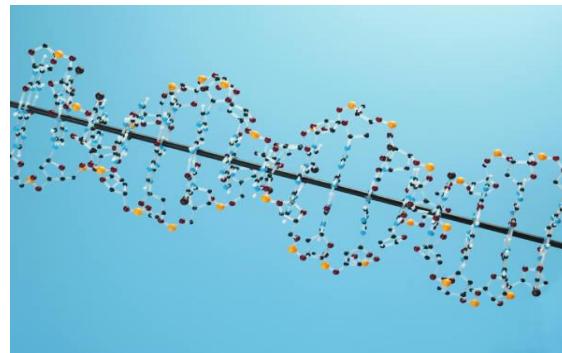
Data transmitted from space probes



Compact discs and DVDs

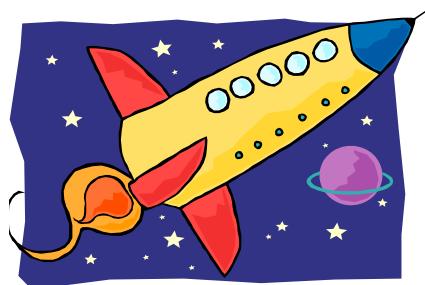


“Reading” DNA



Hamming Error-Correcting Codes

- Binary - use only **0s** and **1s**
- Assumption:
at most one error per codeword
during transmission
- Will **detect** a single error and **correct** it



A Small Hamming Code

Bit Strings of length 3

Codewords: **111** **000**

Correct to 111: **110** **101** **011**

Correct to 000: **001** **010** **100**

Every bit string of length 3 is either

- (i) a codeword or
- (ii) corrects to a codeword

Hamming Code and Hat Game

1 1 1 **Hamming Codeword**

1 1 0

1 0 1

1 0 0

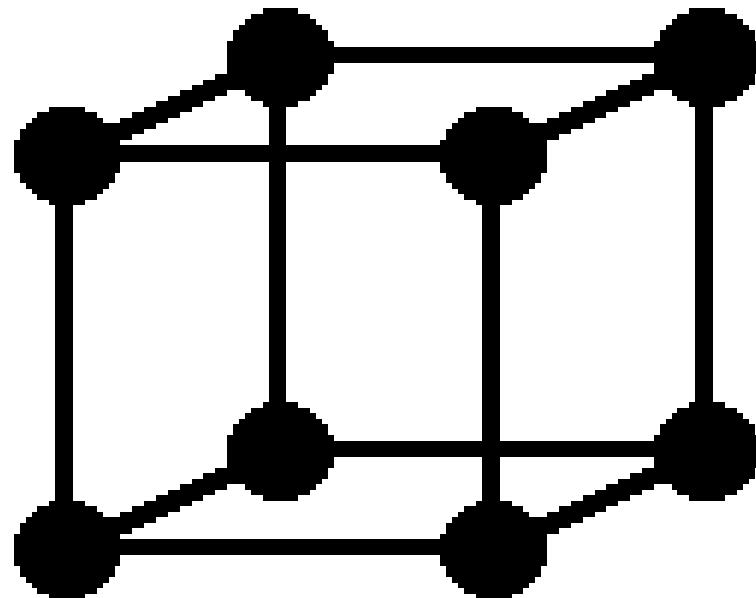
0 1 1

0 1 0

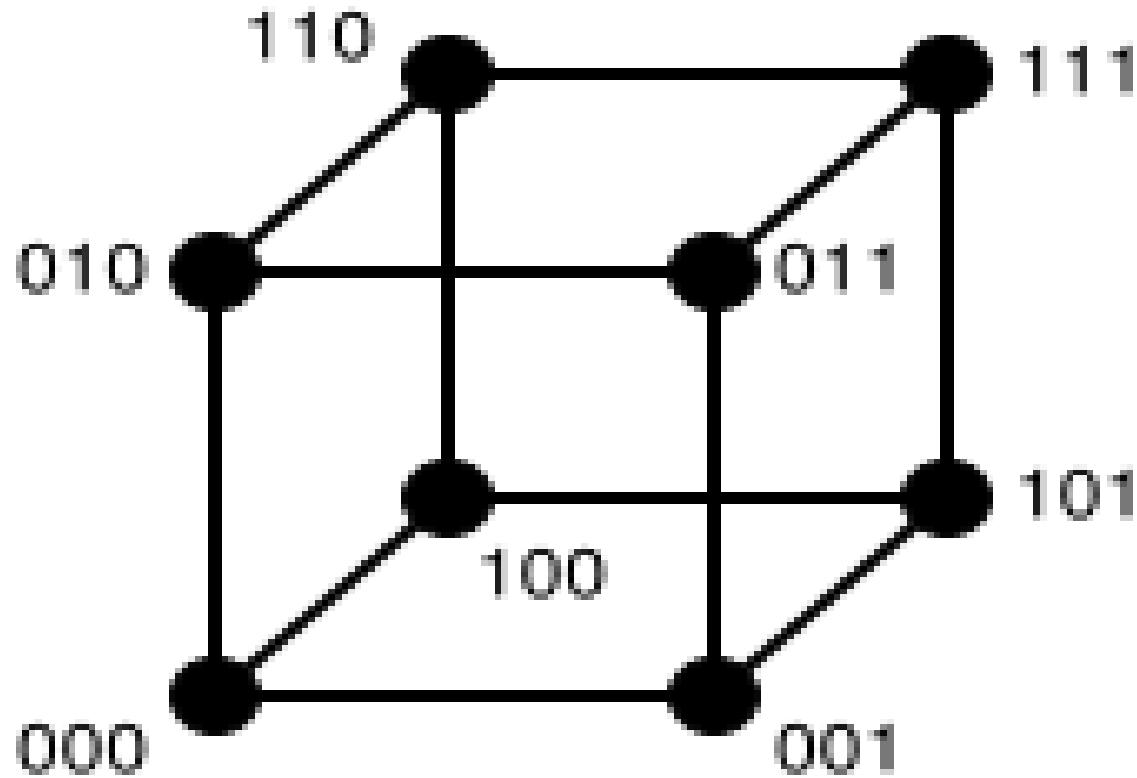
0 0 1

0 0 0 **Hamming Codeword**

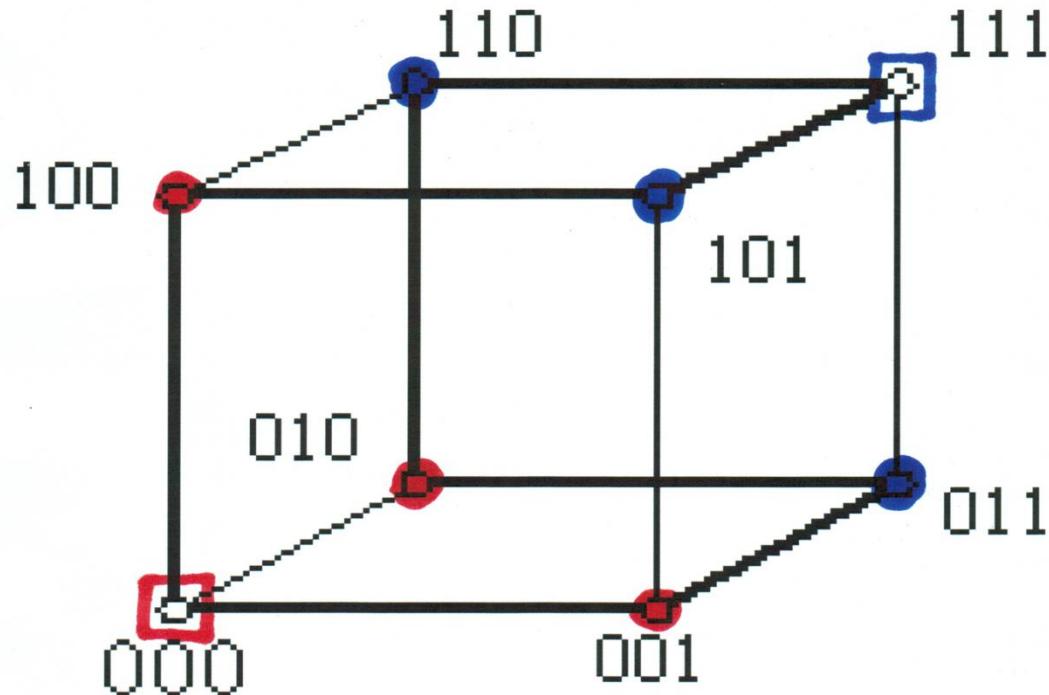
The Cube



Cubes and Bit Strings



Hamming Code on the Cube



A Small Hamming Code

Bit Strings of length 3

Codewords: **111** **000**

Correct to 111: **110** **101** **011**

Correct to 000: **001** **010** **100**

Every bit string of length 3 is either

- (i) a codeword or
- (ii) corrects to a codeword

Larger Hamming Codes

- More than two codewords
- Bit strings must be longer than 3 bits
- Employ *efficient* redundancy
- *Every pair* of codewords must differ in at least 3 positions

11001 and 10010 could both be codewords

10001 and 11000 could NOT both be codewords

Where do larger Hamming Codes *live*?

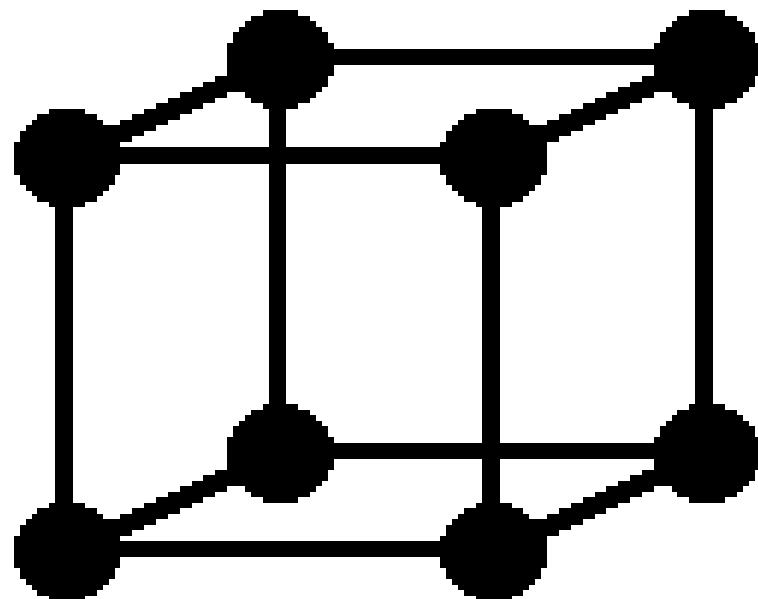
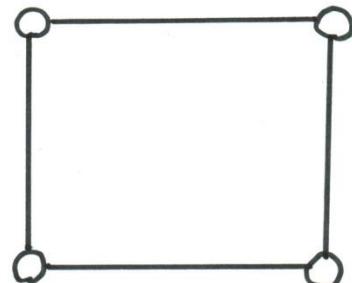
Hamming code with 2 codewords (111 and 000):

lives on the cube

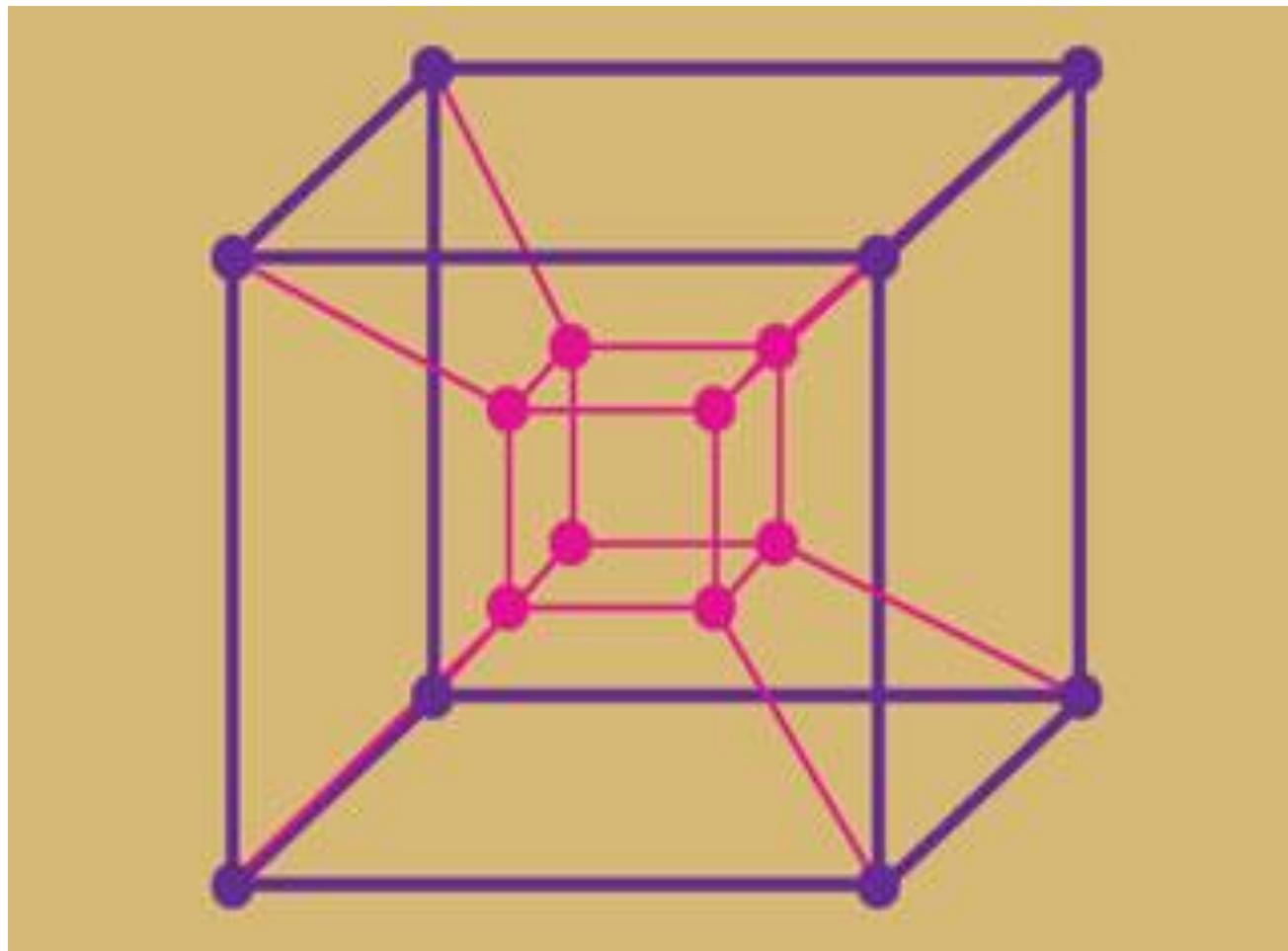
Hamming codes with more than 3 bits per codeword:

we consider a “cubelike” structure that accommodates longer bit strings

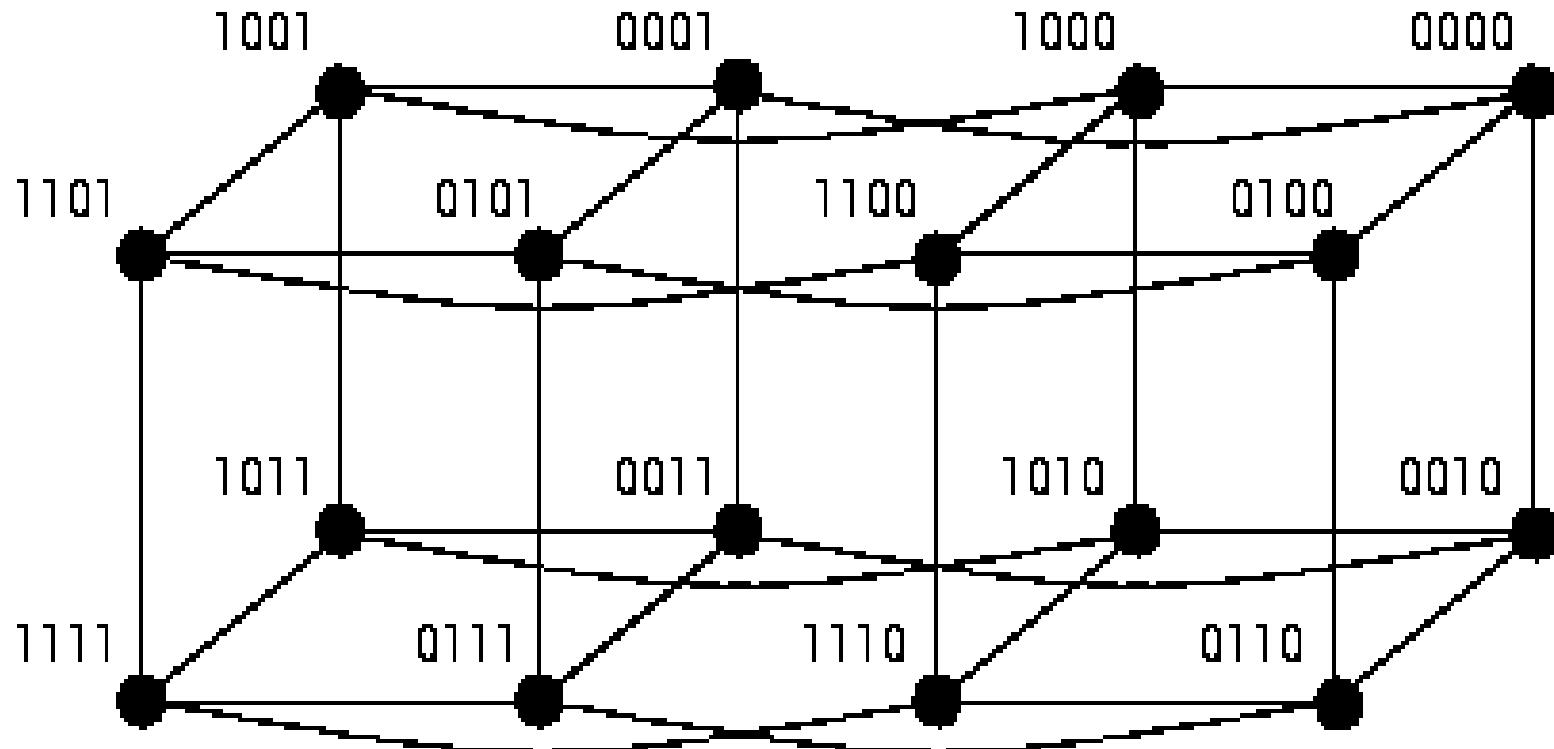
One, Two and Three Dimensional Cubes



The 4D Hypercube



Bit Strings on the 4D Hypercube



Hamming Codewords on Hypercubes

3D cube -- exactly 2 codewords

4D cube -- add a third codeword?

codewords will be bit strings of length 4

Another Hamming Code

- 16 codewords, with each codeword seven bits long

0000000	0001111	0010110	0011001
0100101	0101010	0110011	0111100
1000011	1001100	1010101	1011010
1100110	1101001	1110000	1111111

- $2^7 = 128$ different bit strings that are seven bits long
- Where does this Hamming Code live?

Hamming Code with 7-bit codewords

Because of the 7-bit strings, it will live on the 7D cube.

Building the 7D cube:

connect two 4D cubes for a 5D cube

connect two 5D cubes for a 6D cube

connect two 6D cubes for a 7D cube

The Hat Game with 7 Players

Best strategy -- associated with the Hamming Code that uses 7-bit codewords.

Losses -- occur only with hat configurations that correspond to the 16 codewords.

$2^7 = 128$ possible configurations for 7 hats.

Probability of winning is $\frac{112}{128} = \frac{7}{8}$.

The Hat Game with 7 Players

The best strategy for $N = 7$ players:

- Before the game, each player is assigned a number from 1 to 7.
- Once the hats are in place, each player constructs the column vector X corresponding to the hat configuration that he observes,
 - with “1” = **Blue**, “0” = **Red**, and with a “0” in her own entry.
- Each player then computes $H \cdot X \text{ mod } 2$, where H is the appropriate-sized binary Hat/Hamming matrix (namely, H will have as its columns the binary representations of the numbers 1 through 7).

The Hat/Hamming Matrix for 7 Players

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The 7-Hat Game Best Strategy

If the result of $H \cdot X \bmod 2$ is

- all 0s, the player guesses “1”.
- the *binary expansion of his own number*,
the player guesses “0”.
- Otherwise, the player Passes.

7-Hat Game Example

Suppose the hats are **1101100**.

Player 1:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

7-Hat Game Example

The hats are **1101100**.

Player 1:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} x \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{ mod } 2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

So Player 1 passes.

7-Hat Game Example

The hats are **1101100**.

Player 2:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} x \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \text{ mod } 2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So Player 2 guesses “1”.

7-Hat Game Example

The hats are **1101100**.

Players 3, 6 and 7:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} x \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \text{ mod } 2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

So Players 3, 6 and 7 pass.

7-Hat Game Example

The hats are **1101100**.

Player 4:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} x \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ mod } 2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

So Player 4 passes.

7-Hat Game Example

The hats are **1101100**.

Player 5:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} x \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ mod } 2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

So Player 5 passes.

7-Hat Game Example

If the hats are **1101100**,

Player 2 correctly guesses “1”.

All other players pass.

So the team wins!

Another 7-Hat Game Example

Suppose the hats are **0110011**.

Player 1:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Another 7-Hat Game Example

The hats are **0110011**.

Player 1:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} x \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} \text{ mod } 2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So Player 1 guesses “1”.

Another 7-Hat Game Example

The hats are **0110011**.

Player 6:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} x \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \text{ mod } 2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

So Player 6 guesses “0”.

Another 7-Hat Game Example

If the hats are **0110011**,

Players 1, 4 and 5 get all 0s and guess “1”.

Players 2,3,6 and 7 each get their own number and guess “0”.

So the team loses,

with everyone guessing incorrectly!

Beyond Seven

Hat Game

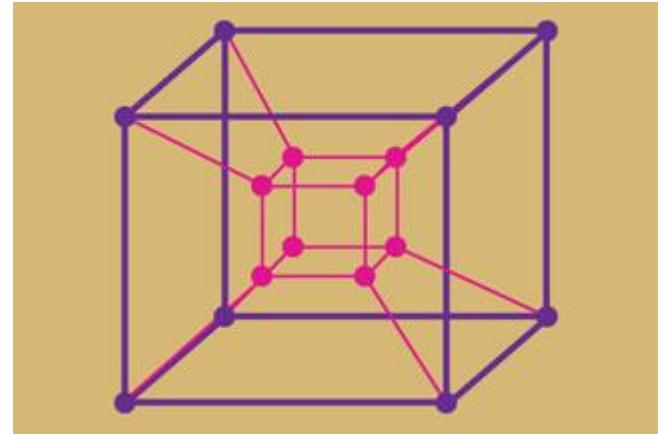
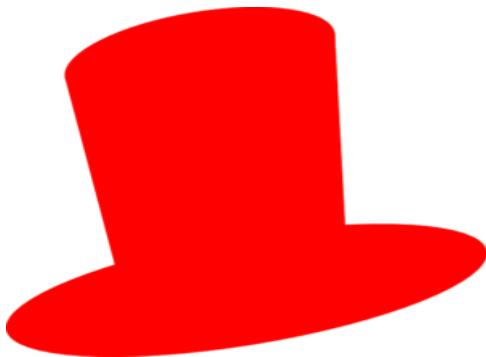
Number of Players	Best strategy Probability of winning
3	$3/4$
7	$7/8$
15	$15/16$
31	$31/32$
63	$63/64$
and so on	

Beyond Seven

Hamming Codes

Length of codewords	Total number of codewords
3 bits	2
7 bits	16
15 bits	2048
31 bits	67,108,864
and so on	

Hats, Hamming and Hypercubes



Employing H, H and H with students

3-Hat Game

- Can be used stand alone, with no prerequisites
- Gen Ed, Liberal Arts, QR, Math Ed courses
 - Connect to truth tables if desired
- First-Year Seminar
- Critical Thinking Game of Strategy
- Math Clubs or Math Circles
- Also with faculty groups

Employing H, H and H with students

Hamming Codes

- Can be used stand alone, with some understanding of binary numbers
- Gen Ed, Liberal Arts, QR, Math Ed courses
 - Connect to truth tables if desired
- Connects well with error-detecting codes
 - UPC, ISBN, and many others
- Math Clubs or Math Circles
- Discrete Math or Combinatorics

Employing H, H and H with students

Hypercubes

- Can be used stand alone, with no prerequisites
- Gen Ed, Liberal Arts, QR, Math Ed courses
- First-Year Seminar
 - Tesseract
- Math Clubs or Math Circles
- Computer Science applications
- Discrete Math, Graph Theory or Combinatorics

Resources

Bernstein, Mira. “The Hat Problem and Hamming Codes” in *Focus: The Newsletter of the Mathematical Association of America*, November 2001, pp. 4-6.

Roman, Steven. *Codes and Coding, Third Edition, Modules in Mathematics*. Innovative Textbooks, Irvine, CA.