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Shortest paths Puzzle Steiner problem Solution Generalizing Soap films

Minimal surfaces

# Shortest paths, soap films, and mathematics

Joint Mathematics Meetings, January 2015

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### Puzzle

The shortest path connecting these two points is

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### Puzzle

The shortest path connecting these two points is



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## Puzzle

The shortest path connecting these two points is

What is the shortest path connecting these four points?



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What is the correct answer?

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### Steiner Problem

What is the shortest path connecting these four points?

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### Steiner Problem

What is the shortest path connecting these four points?

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Let  $f(x) = x + 4\left(\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1-x}{2}\right)^2}\right)$ 

$$= x + 2\sqrt{1 + (1 - x)^2}.$$

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Then 
$$0 = f'(x) = 1 + \frac{-2(1-x)}{\sqrt{1+(1-x)^2}}$$

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Then 
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So,  $Length = \left(1 - \frac{1}{\sqrt{3}}\right) + 2\left(\sqrt{1 + \frac{1}{3}}\right)$ 

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So, 
$$Length = \left(1 - \frac{1}{\sqrt{3}}\right) + 2\left(\sqrt{1 + \frac{1}{3}}\right) = 1 + \sqrt{3} \approx 2.73.$$

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Note that 
$$\sin \theta = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{1+\frac{1}{3}}} = \frac{\sqrt{3}}{2}$$

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$$\sin \theta = rac{rac{1}{2}}{rac{1}{2}\sqrt{1+rac{1}{3}}} = rac{\sqrt{3}}{2} \; \Rightarrow \; \theta = 60^\circ.$$

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Therefore, the lines meet at

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So, 
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Note that 
$$\sin \theta = rac{rac{1}{2}}{rac{1}{2}\sqrt{1+rac{1}{3}}} = rac{\sqrt{3}}{2} \; \Rightarrow \; \theta = 60^\circ.$$

Therefore, the lines meet at  $120^{\circ}$  angles.

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(1) Use more points:

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(1) Use more points: What is the shortest path connecting n points?

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(1) Use more points: What is the shortest path connecting n points?



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(1) Use more points: What is the shortest path connecting n points?



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Steiner Problem:

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Steiner Problem:

minimizes distance

in a plane

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Steiner Problem:

minimizes distance (1-d object) in a plane (2-d world)

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#### Steiner Problem:

# minimizes distancein a plane(1-d object)(2-d world)↓↓(2-d object)(3-d world)

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#### Steiner Problem:

minimizes distance (1-d object) ↓ (2-d object) minimizes area in a plane (2-d world) ↓ (3-d world) in space

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#### Steiner Problem:



in a plane (2-d world) ↓ (3-d world) in space

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What does this?

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#### Steiner Problem:



What does this? soap films and minimal surfaces

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# Soap Films

# Soap films model surfaces that minimize area in space.

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Soap films

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# Soap Films

# Soap films model surfaces that minimize area in space.

# Let's model some minimal surfaces!

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Soap films

minimize area locally

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Minimal surfaces

- minimize area locally
- look like saddle surfaces,

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Minimal surfaces

- minimize area locally
- look like saddle surfaces,
  - at each point, the bending upward in one direction

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- minimize area locally
- look like saddle surfaces,
  - at each point, the bending upward in one direction is matched with the bending downward in the orthogonal direction.

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- minimize area locally
- look like saddle surfaces,
  - at each point, the bending upward in one direction is matched with the bending downward in the orthogonal direction.

#### Examples





Scherk doubly-periodic

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Enneper



helicoid

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Enneper



Twisted Scherk



helicoid



Costa-Hoffman-Meeks

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Thank you!

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# Thank you!

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