Shortest paths, soap films, and mathematics

Joint Mathematics Meetings, January 2015

Michael Dorff
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Puzzle

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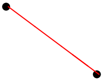
The shortest path connecting these two points is 

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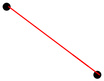
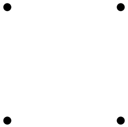
The shortest path connecting these two points is 

What is the shortest path connecting these four points?



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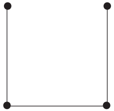

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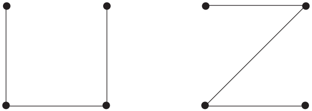

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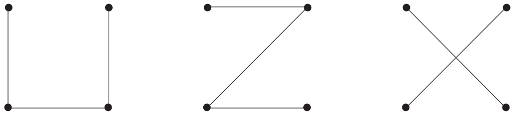

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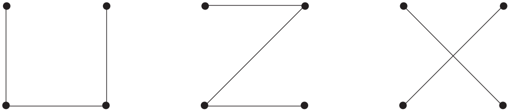

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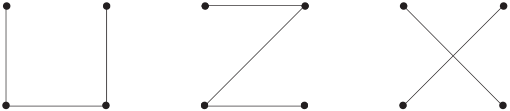

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What is the correct answer?

Steiner Problem

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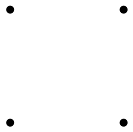

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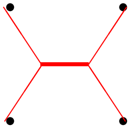
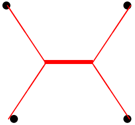
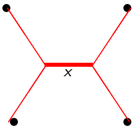
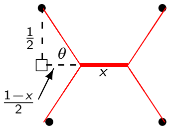





x


$\frac{1}{2}$
θ
x
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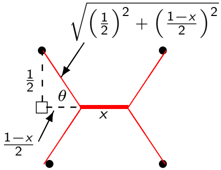
$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1-x}{2}\right)^2}$
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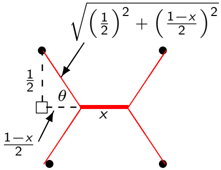

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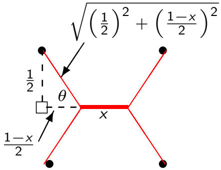

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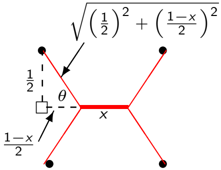

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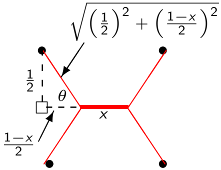

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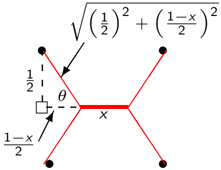

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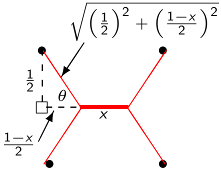

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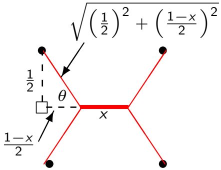

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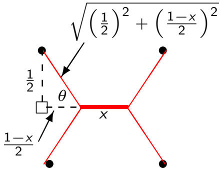

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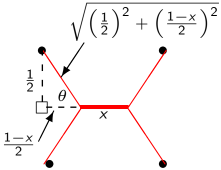

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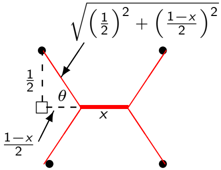

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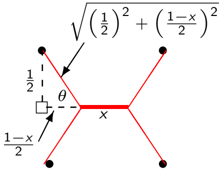

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Therefore, the lines meet at 120° angles.

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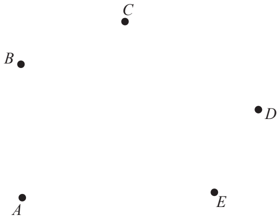

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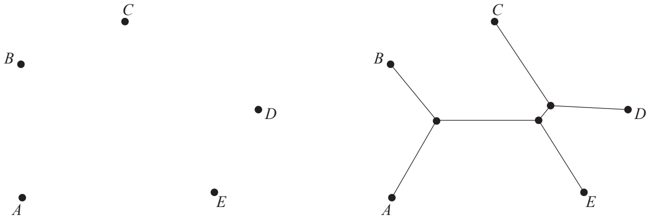

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⇓	⇓
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$\Downarrow$	$\Downarrow$
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What does this? soap films and minimal surfaces

Soap Films

Soap films model surfaces that minimize area in space.

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Let's model some minimal surfaces!

Minimal surfaces in $\mathbb{R}^3$

- minimize area locally

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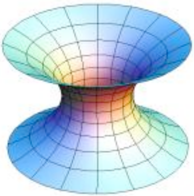
Examples

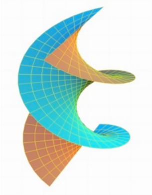


catenoid

Scherk doubly-periodic

Examples

Enneper



helicoid

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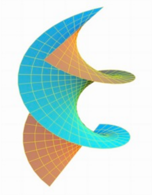

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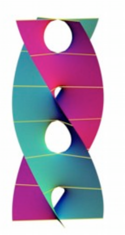
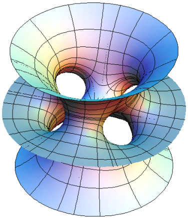


Twisted Scherk



Costa-Hoffman-Meeks

Thank you!

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mdorff@math.byu.edu