Example 1

Find x_1, x_2, x_3 so that

$$x_1 \underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{\vec{v}_1} + x_2 \underbrace{\begin{bmatrix}1\\-2\\1\end{bmatrix}}_{\vec{v}_2} + x_3 \underbrace{\begin{bmatrix}1\\0\\-1\end{bmatrix}}_{\vec{v}_3} = \underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix}}_{\vec{b}}$$

i.e.

$$\begin{bmatrix}1&1&1\\1&-2&0\\1&1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

GE

$$\left[\begin{array}{ccc|c}1&1&1&1\\1&-2&0&2\\1&1&-1&3\end{array}\right] \sim \left[\begin{array}{ccc|c}1&0&0&2\\0&1&0&0\\0&0&1&-1\end{array}\right]$$

AI

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1&1&1\\1&-2&0\\1&1&-1\end{bmatrix}^{-1}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}2\\0\\1\end{bmatrix}$$

LS

$$A^TA = \begin{bmatrix}3&0&0\\0&6&0\\0&0&2\end{bmatrix} \qquad A^T\vec{b} = \begin{bmatrix}6\\0\\-2\end{bmatrix}$$

so

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}3&0&0\\0&6&0\\0&0&2\end{bmatrix}^{-1}\begin{bmatrix}6\\0\\-2\end{bmatrix} = \begin{bmatrix}2\\0\\-1\end{bmatrix}.$$

OP

$$\vec{b} = \frac{\vec{b}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{b}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 + \frac{\vec{b}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3}\vec{v}_3$$

$$= \frac{6}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{0}{6}\begin{bmatrix}1\\-2\\1\end{bmatrix} + \frac{-2}{2}\begin{bmatrix}1\\0\\-1\end{bmatrix}.$$

Three advantages of OP: Easier
Can find x_1, x_2 or x_3 — don't need all 3.
Don't need a complete set of vectors onto which to project.

Example 2

LS $\quad \hat{\vec{x}} = (A^T A)^{-1}(A^T \vec{b})$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

OP $\quad \vec{b} = \dfrac{\vec{b} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \dfrac{\vec{b} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$

$$= \frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{0}{6} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Example 3

GE $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$

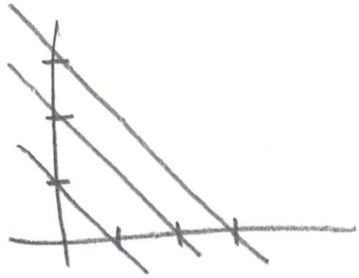
LS $\hat{\vec{x}} = (A^T A)^{-1}(A^T \vec{b})$

$= \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$

Example 4

$x + y = 1$
$x + y = 2$
$x + y = 3$



$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

<u>LS</u> Solve for $\hat{\vec{x}}$ in $A^T A \hat{\vec{x}} = A^T \vec{b}$

($A^T A$ does not have an inverse.)

+ <u>GE</u> $\left[\begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 3 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$

So any x, y such that $x + y = 2$

(any point on the middle line).

Why not just GE from the start?

There are ∞ "best" solutions.

Example 5

$$\begin{aligned} x + 2y &= 3 \\ 2x + 4y &= 6 \\ 3x + 6y &= 9 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{array}\right]$$

GE $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

so any x, y such that $x + 2y = 3$.

LS Solve for $\hat{\vec{x}}$ in $A^T A \hat{\vec{x}} = A^T \vec{b}$:

+ GE $\left[\begin{array}{cc|c} 14 & 28 & 42 \\ 28 & 56 & 84 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right]$

so any x, y such that $x + 2y = 3$.

GE works because there is (there are) exact solutions.

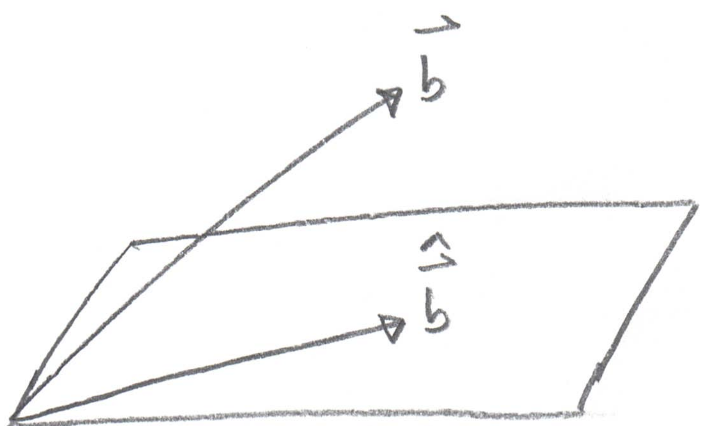
Same solutions either way.

Example 6   rank $A = 2$, rank $A^T A = 2$

LS   Solve for $\hat{\vec{x}}$ in $A^T A \hat{\vec{x}} = A^T \vec{b}$

+ GE

$$\left[\begin{array}{ccc|c} 14 & 32 & 50 & 6 \\ 32 & 77 & 122 & 12 \\ 50 & 122 & 194 & 18 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 13/9 \\ 0 & 1 & 2 & -4/9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$



$$\hat{\vec{b}} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

∞ solutions, none are exact,
all are best.

Example 7 rank $A = 2$, rank $A^TA = 2$

LS Solve for $\hat{\vec{x}}$ in $A^TA\hat{\vec{x}} = A^T\vec{b}$

+ GE
$$\left[\begin{array}{cccc|c} 14 & 32 & 50 & 6 & 46 \\ 32 & 77 & 122 & 15 & 109 \\ 50 & 122 & 194 & 24 & 172 \\ 6 & 15 & 24 & 3 & 21 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1/3 & 1 \\ 0 & 1 & 2 & 1/3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Same picture as for Example 6.

Example 8 $\quad$ rank $A = 3$, rank $A^TA = 3$

GE

$$\left[\begin{array}{cccc|c} 1 & 4 & 7 & 1 & 2 \\ 2 & 5 & 8 & 1 & 1 \\ 3 & 6 & 0 & 1 & -9 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/3 & -1 \\ 0 & 1 & 0 & 1/3 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

LS $\quad$ Solve for $\hat{\vec{x}}$ in $A^TA\hat{\vec{x}} = A^T\vec{b}$

+ GE

$$\left[\begin{array}{cccc|c} 14 & 32 & 23 & 6 & -23 \\ 32 & 77 & 68 & 15 & -41 \\ 23 & 68 & 113 & 15 & 22 \\ 6 & 15 & 15 & 3 & -6 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/3 & -1 \\ 0 & 1 & 0 & 1/3 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Same solutions either way.