Example:

~~Suppose~~ $A = \begin{bmatrix} 2 & 3 \\ -2 & 7 \end{bmatrix}$.

E-values of A: values of λ for which

$A\vec{v} = \lambda\vec{v}$, i.e. $\underbrace{(A - \lambda I)}_{\text{Bad matrix, so } \det(A-\lambda I) = 0}\vec{v} = \vec{0}$.

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ -2 & 7-\lambda \end{vmatrix}$$

$$= \cdots = (\lambda - 4)(\lambda - 5)$$

so $\lambda_1 = 4$ or $\lambda_2 = 5$.

E-vectors:

$\lambda = 4$: $A - 4I = \begin{bmatrix} -2 & 3 \\ -2 & 3 \end{bmatrix}$

$$\left[\begin{array}{cc|c} -2 & 3 & 0 \\ -2 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 - \tfrac{3}{2}x_2 = 0$$

$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix},$$

so any multiple of $\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$,

i.e. " " " $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Call this $\vec{v}_1$.

Similarly, for $\lambda = 5$, e-vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftarrow \vec{v}_2$.

$\vec{v}_1$ and $\vec{v}_2$ are linearly independent, which was guaranteed, since $\lambda_1 \neq \lambda_2$, so form a basis for $\mathbb{R}^2$ (since there are <u>two</u> vectors).

Given arbitrary vector, say $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \cdots = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

So $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} - 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} + \begin{bmatrix} -5 \\ -5 \end{bmatrix}$

So for basis $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$,

$$[\vec{x}]_{\mathcal{B}} = (2, -5) = \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

Then $A^k \vec{x} = A^k (c_1 \vec{v_1} + c_2 \vec{v_2})$

$$= \lambda_1^k c_1 \vec{v_1} + \lambda_2^k c_2 \vec{v_2}$$

$$= 4^k \begin{bmatrix} 6 \\ 4 \end{bmatrix} + 5^k \begin{bmatrix} -5 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \cdot 4^k - 5 \cdot 5^k \\ 4 \cdot 4^k - 5 \cdot 5^k \end{bmatrix}$$

and $A = PDP^{-1}$, $A^k = PD^kP^{-1}$

where $\underbrace{P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}}_{\text{or } P = \begin{bmatrix} 6 & -5 \\ 4 & -5 \end{bmatrix}}$, $D = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$

So $A^k = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4^k & 0 \\ 0 & 5^k \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$

$$= \cdots = \underbrace{\begin{bmatrix} 3 \cdot 4^k - 2 \cdot 5^k & -3 \cdot 4^k + 3 \cdot 5^k \\ 2 \cdot 4^k - 2 \cdot 5^k & -2 \cdot 4^k + 3 \cdot 5^k \end{bmatrix}}_{\text{a 2X2 matrix}}$$

So $A^k \vec{x} = \begin{bmatrix} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 6 \cdot 4^k - 5 \cdot 5^k \\ 4 \cdot 4^k - 5 \cdot 5^k \end{bmatrix}}_{\text{a vector}}$

For the matrix A above, $\lim\limits_{k\to\infty} A^k$ is infinite.

Suppose A is a stochastic/probability matrix,

say $A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$, and

$$\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = .7 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

e-values

e-vectors

Then $A^k = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1^k & 0 \\ 0 & .7^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$

and $A^{\infty} = \lim\limits_{k\to\infty} A^k = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$

since $\lim\limits_{k\to\infty} .7^k = 0$

$$= \dots = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

The equilibrium vector with sum of entries 1

(so it is a probability vector)

This is also the e-vector for e-value of 1.