10 Examples in Linear Algebra
Math 260

Linear Algebra is one of the most interesting areas in all of mathematics, and perhaps more importantly, one of the most useful areas. There are literally thousands of different applications of the ideas and techniques we will discuss and develop in this course. We'll see a just few of these applications this semester. To give you some idea of what Linear Algebra is good for, as well as to create some continuity in this course, we will regularly refer to the following ten examples at various times throughout the semester.

1. Investments
2. Medical Imaging
3. Balancing Chemical Equations
4. Traffic Flow
5. Nutrition
6. Markov Chains
7. Input-out Model
8. Predator-Prey Models
9. Data Fitting
10. Frequencies in Music, Images and Other Signals; Fourier Series

Investments (no particular section)

Background: It is generally a bad idea to put all of your money in one investment. It is generally a good idea to diversify. Given a collection of investments and a certain amount of money to invest, the question is how much to allocate to each investment. Of course, in the real world, how to do this is complicated and full of risk, and a lot of people make a lot of money trying to convince other people that they know how best to do this. We will look at a very simple version of this problem.

Problem: A bank wishes to invest a $100,000 trust fund in three sources: bonds paying an 8% annual return, certificates of deposit (CDs) paying 7%, and mortgages paying 10%. The bank wishes to realize an $8000 annual income from the investment. One condition of the trust is that the total amount invested in bonds and CDs must be triple the amount invested in mortgages. How much should be invested in each category to realize the desired $8000 return per year?

Medical Imaging (no particular section)

Background: A variety of scanning techniques, such as CT scans, CAT scans, PET scans, MRIs, etc., are used to create a 2D or 3D image or picture of a certain part of the body, such as the brain or the heart. We will consider the simplest possible version of these scans.

Problem: As illustrated below for the simplest 2D case, a signal/ray is emitted and passes through the body and detected on the other side. The decrease in the signal/ray corresponds to the type of material (bone, blood, tumor, etc.) through which the signal/ray has passed, and the material can thus be deduced by the amount of decrease in the signal/ray.

$$
\begin{array}{cc|c|c|cc}
 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\text{Ray}} & \multicolumn{1}{c}{\text{Ray}} & & \\
 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\downarrow} & \multicolumn{1}{c}{\downarrow} & & \\
\cline{3-4}
\text{Ray} & \rightarrow & m_1 & m_2 & \rightarrow & s_1 \\
\cline{3-4}
\text{Ray} & \rightarrow & m_3 & m_4 & \rightarrow & s_2 \\
\cline{3-4}
 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\downarrow} & \multicolumn{1}{c}{\downarrow} & & \\
 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{s_3} & \multicolumn{1}{c}{s_4} & &
\end{array}
$$

Given four readings s_1, s_2, s_3 and s_4 from the four *sensors*, determine the values m_1, m_2, m_3 and m_4 at the four locations in the image (which would allow us to deduce the *material* in each of the four locations).

Balancing Chemical Equations (Section 1.6)

Background: Chemical equations describe the quantities of substances consumed and produced by chemical reactions. Chemists are interested in what amounts of each substance involved must be present for the reaction to occur.

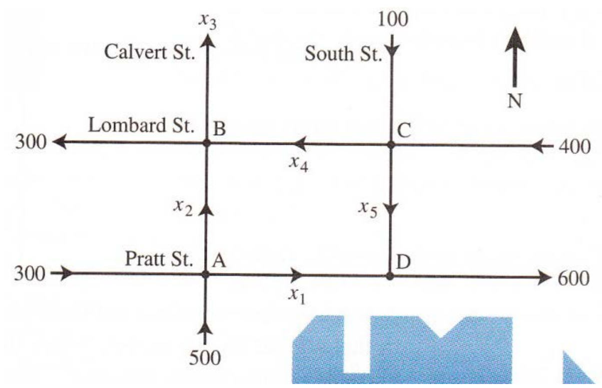
Problem: When propane burns, the propane C_3H_8 combines with oxygen O_2 to form carbon dioxide CO_2 and water H_2O, according to the form:

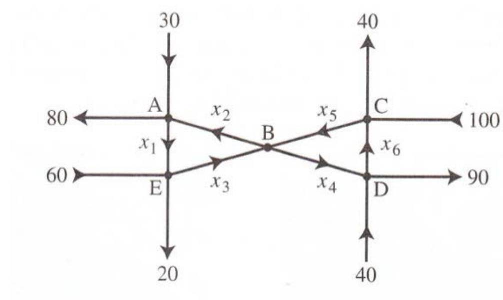
$$a_1 \cdot C_3H_8 + a_2 \cdot O_2 \rightarrow a_3 \cdot CO_2 + a_4 \cdot H_2O$$

What amounts a_1, a_2, a_3, a_4 of each substance would be needed to balance the above process?

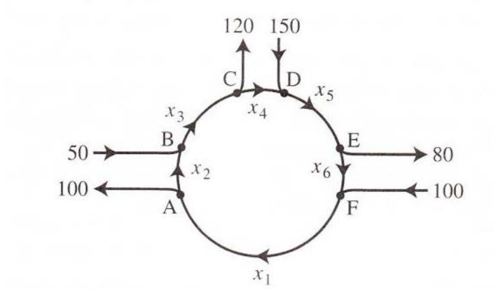
Traffic Flow (Section 1.6)

Background: Traffic congestion is a serious problem. For example, a lot of time is wasted and accidents occur because of too much traffic trying to move through an intersection that cannot handle the given amount of traffic. To make things more interesting, of course, there are all sorts of types of intersections, such as those shown below.





Problem: Intersections in Italy are often constructed in one-way *circoli* (which literally means *circles*)—what we in America call roundabouts or traffic circles—such as the one below.



Assuming that traffic must flow in the directions shown, find the general solution of the traffic flow.

Nutrition (Section 1.10)

Background: Dieticians often try to combine different foods to supply a particular amount of each of several nutrients. The question is how much of each type of food to supply exact the desired amounts?

Problem: Using three foods, a dietician wants to supply a client with exactly 1000 units of Vitamin A, 1600 units of Vitamin C and 2400 units of Vitamin E. The vitamin content of each food is list below:

One gram of Food One contains 2 units of Vitamin A, 3 units of Vitamin C, and 5 units of Vitamin E.

One gram of Food Two contains 4 units of Vitamin A, 7 units of Vitamin C, and 9 units of Vitamin E.

One gram of Food Three contains 6 units of Vitamin A, 10 units of Vitamin C, and 14 units of Vitamin E.

How much of each food should be eaten to end up with exactly the desired amounts of the vitamins?
Same question, but with a desired amount of 2700 units of Vitamin E.
Same questions, but if Food One contains just 1 unit of Vitamin A per gram.

Markov Chains (Section 1.10, Section 4.9)

Background: There are a number of situations in which a group of people (or other things) can be divided into two or more categories and which after regular periods of time change which categories they are in. For example, of current Coke and Pepsi drinkers, some will still prefer the same drink after a year and some will have switched preferences to the other drink, and we are interested in the long term behavior (after five or ten or twenty years) of the drink preferences of these people. There are all sorts of applications of Markov Chains ranging from economics and finance to children's board games (like Chutes and Ladders) to the possible outcome of a single inning in baseball.

Problem: Suppose that of those who currently live in the city and suburbs, after one more year:

70% of those in the city stay there, and 30% move to the suburbs.

50% of those in the suburbs remain there and 50% move to the city.

In the long run, what percentage of the population will be in the city and what percentage will be in the suburbs (or perhaps will these percentages simply fluctuate up and down over and over)?

Input-output Model (Section 2.6)

Background: The world is connected in so many ways. One such way is how countries buy/sell good from/to each other. As business is done, usually producing some of one item (e.g. electricity) consumes some of another (e.g. coal), as well as possibly some of itself. Thus to sell a unit of electricity to another country, one must produce more than the unit desired (since it consumes a bit of electricity in the process) as well as a bit of the other interdependent goods like coal (since it might be burned to create the electricity). This sounds pretty simple, and in a way it is, but on the global scale it becomes very complicated. And understanding how this all works is very important to economists and for economies to function efficiently. In fact, the Russian-American economist Wassily Leontief won the Nobel Prize in Economics in 1973 for the mathematical depth and the real world usefulness of the work he did in this area.

Problem: Suppose that producing one unit of coal consumes .4 units of electricity (e.g. to run the machines the dig the coal, to run the lights, etc.) and producing one unit of electricity consumes .2 units of coal (which is burned to produce the electricity) and .1 units of electricity (to run the lights, machines, etc.). How much coal and electricity should we produce, knowing that some of each will be consumed in the process, in order to end up with 10 units each of coal and electricity?

Predator-Prey Models (Section 5.6, Section 5.7)

Background: The are all sorts of dynamical systems in the world in which different quantities are interdependent and which change as time goes by. (*Dynamic* or *dynamical* simply means *changing*.) Examples of dynamical systems range from investments to the so-called Butterfly Effect to animal population sizes when some of the animals eat the others. A fairly simple and interesting dynamical system is the Predator-Prey Model, in which there is a single predator (the one eating) and a single prey (the one being eaten).

Problem: Where O_k is the number of owls after k months and R_k is the number of rats (in thousands) after k months, we have:

$$\begin{aligned} O_{k+1} &= \ .5\ O_k + \ .4\, R_k \\ R_{k+1} &= -p \cdot O_k + 1.1\, R_k \end{aligned}$$

where p is a positive parameter which essentially describes how quickly the owls eat the rats.

What is the long term behavior of the owl and rat populations? Will one eventually die out? What effect does increasing or decreasing the parameter p have on the problem?

Data Fitting (Section 6.5)

Background: A common task of mathematicians and scientists from all disciplines is to fit a straight (or curvy) line to some given data, both to better understand the data as well as to make estimates or predictions based on the data. Another fairly common situation is that there are too many restrictions to find an exact solution to a linear system of equations, and we do the best we can in finding the solution that best satisfies the given conditions as well as possible. The problem we will look at simply consists of finding a straight line that describes how lung cancer death rates depend on smoking rates, given just four data points. (More often, we have dozens or hundreds or thousands of data points.)

Problem: The following table gives the per capita consumption of cigarettes in 1930 in four countries and the male death rate for lung cancer in 1950.

Country	Cigarette Consumption (per capita)	Lung Cancer Deaths (per million males)
Norway	250	95
Sweden	300	120
Denmark	350	165
Australia	470	170

Find the straight line that best fits these data. What is this line useful for?

Frequencies in Music, Images and Other Signals; Fourier Series (Section 6.8)

Background: There are various signals that are composed of a plethora (don't you just love that word) of frequencies. A single note from a piano or a trumpet might be the same note, but for some reason they sound different. This is because of the harmonics, overtones, undertones, and in general the various frequencies other than the actual note you are hearing, that are present in the note played by each instrument. Digital (and analog) images and other signals are similar. It turns out that this understanding is hugely useful in today's world for building synthesizers, compressing images, and a thousand other applications. We will look at Fourier Series, which is about as simple (and extremely interesting mathematically) a version of this sort of problem as there is.

Problem: Given a function, we want to build it out of sines and cosines. For example, we can show

$$t = 2(\sin t - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \frac{\sin 4t}{4} + \frac{5 \sin 5t}{5} - \cdots)$$

Two questions: how to we come up with this representation (decomposition) of the function t into these various frequencies of sine and what is it useful for?