6.7 SOLUTIONS

Notes: The three types of inner products described here (in Examples 1, 2, and 7) are matched by examples in Section 6.8. It is possible to spend just one day on selected portions of both sections. Example 1 matches the weighted least squares in Section 6.8. Examples 2–6 are applied to trend analysis in Seciton 6.8. This material is aimed at students who have not had much calculus or who intend to take more than one course in statistics.

For students who have seen some calculus, Example 7 is needed to develop the Fourier series in Section 6.8. Example 8 is used to motivate the inner product on $C[a, b]$. The Cauchy-Schwarz and triangle inequalities are not used here, but they should be part of the training of every mathematics student.

1. The inner product is $\langle x, y\rangle = 4x_1y_1 + 5x_2y_2$. Let $\mathbf{x} = (1, 1)$, $\mathbf{y} = (5, -1)$.
   a. Since $\|\mathbf{x}\|^2 = \langle x, x\rangle = 9$, $\|\mathbf{x}\| = 3$. Since $\|\mathbf{y}\|^2 = \langle y, y\rangle = 105$, $\|\mathbf{y}\| = \sqrt{105}$. Finally, $|\langle x, y\rangle|^2 = 15^2 = 225$.
   b. A vector $\mathbf{z}$ is orthogonal to $\mathbf{y}$ if and only if $\langle x, y\rangle = 0$, that is, $20z_1 - 5z_2 = 0$, or $4z_1 = z_2$. Thus all multiples of $\begin{bmatrix}1\\4\end{bmatrix}$ are orthogonal to $\mathbf{y}$.

2. The inner product is $\langle x, y\rangle = 4x_1y_1 + 5x_2y_2$. Let $\mathbf{x} = (3, -2)$, $\mathbf{y} = (-2, 1)$. Compute that $\|\mathbf{x}\|^2 = \langle x, x\rangle = 56$, $\|\mathbf{y}\|^2 = \langle y, y\rangle = 21$, $\|\mathbf{x}\|^2\|\mathbf{y}\|^2 = 56\cdot 21 = 1176$, $\langle x, y\rangle = -34$, and $|\langle x, y\rangle|^2 = 1156$. Thus $|\langle x, y\rangle|^2 \le \|\mathbf{x}\|^2\|\mathbf{y}\|^2$, as the Cauchy-Schwarz inequality predicts.

3. The inner product is $\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$, so $\langle 4+t, 5-4t^2\rangle = 3(1) + 4(5) + 5(1) = 28$.

4. The inner product is $\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$, so $\langle 3t - t^2, 3 + 2t^2\rangle = (-4)(5) + 0(3) + 2(5) = -10$.

5. The inner product is $\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$, so $\langle p, p\rangle = \langle 4+t, 4+t\rangle = 3^2 + 4^2 + 5^2 = 50$ and $\|p\| = \sqrt{\langle p, p\rangle} = \sqrt{50} = 5\sqrt{2}$. Likewise $\langle q, q\rangle = \langle 5-4t^2, 5-4t^2\rangle = 1^2 + 5^2 + 1^2 = 27$ and $\|q\| = \sqrt{\langle q, q\rangle} = \sqrt{27} = 3\sqrt{3}$.

6. The inner product is $\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$, so $\langle p, p\rangle = \langle 3t - t^2, 3t - t^2\rangle = (-4)^2 + 0^2 + 2^2 = 20$ and $\|p\| = \sqrt{\langle p, p\rangle} = \sqrt{20} = 2\sqrt{5}$. Likewise $\langle q, q\rangle = \langle 3 + 2t^2, 3 + 2t^2\rangle = 5^2 + 3^2 + 5^2 = 59$ and $\|q\| = \sqrt{\langle q, q\rangle} = \sqrt{59}$.

7. The orthogonal projection $\hat{q}$ of q onto the subspace spanned by p is

$$\hat{q} = \frac{\langle q, p\rangle}{\langle p, p\rangle}p = \frac{28}{50}(4 + t) = \frac{56}{25} + \frac{14}{25}t$$

8. The orthogonal projection $\hat{q}$ of q onto the subspace spanned by p is

$$\hat{q} = \frac{\langle q, p\rangle}{\langle p, p\rangle}p = -\frac{10}{20}(3t - t^2) = -\frac{3}{2}t + \frac{1}{2}t^2$$

9. The inner product is $\langle p, q\rangle = p(-3)q(-3) + p(-1)q(-1) + p(1)q(1) + p(3)q(3)$.
   a. The orthogonal projection $\hat{p}_2$ of p_2 onto the subspace spanned by p_0 and p_1 is
   $$\hat{p}_2 = \frac{\langle p_2, p_0\rangle}{\langle p_0, p_0\rangle}p_0 + \frac{\langle p_2, p_1\rangle}{\langle p_1, p_1\rangle}p_1 = \frac{20}{4}(1) + \frac{0}{20}t = 5$$
   b. The vector $q = p_2 - \hat{p}_2 = t^2 - 5$ will be orthogonal to both p_0 and p_1 and $\{p_0, p_1, q\}$ will be an orthogonal basis for Span$\{p_0, p_1, p_2\}$. The vector of values for q at $(-3, -1, 1, 3)$ is $(4, -4, -4, 4)$, so scaling by 1/4 yields the new vector $q = (1/4)(t^2 - 5)$.

10. The best approximation to $p = t^3$ by vectors in $W = \text{Span}\{p_0, p_1, q\}$ will be

$$\hat{p} = \text{proj}_W p = \frac{\langle p, p_0\rangle}{\langle p_0, p_0\rangle}p_0 + \frac{\langle p, p_1\rangle}{\langle p_1, p_1\rangle}p_1 + \frac{\langle p, q\rangle}{\langle q, q\rangle}q = \frac{0}{4}(1) + \frac{164}{20}(t) + \frac{0}{4}\left(\frac{t^2 - 5}{4}\right) = \frac{41}{5}t$$

11. The orthogonal projection of $p = t^3$ onto $W = \text{Span}\{p_0, p_1, p_2\}$ will be

$$\hat{p} = \text{proj}_W p = \frac{\langle p, p_0\rangle}{\langle p_0, p_0\rangle}p_0 + \frac{\langle p, p_1\rangle}{\langle p_1, p_1\rangle}p_1 + \frac{\langle p, p_2\rangle}{\langle p_2, p_2\rangle}p_2 = \frac{0}{5}(1) + \frac{34}{10}(t) + \frac{0}{14}(t^2 - 2) = \frac{17}{5}t$$

12. Let $W = \text{Span}\{p_0, p_1, p_2\}$. The vector $p_3 = p - \text{proj}_W p = t^3 - (17/5)t$ will make $\{p_0, p_1, p_2, p_3\}$ an orthogonal basis for the subspace $\mathbb{P}_3$ of $\mathbb{P}_4$. The vector of values for p_3 at $(-2, -1, 0, 1, 2)$ is $(-6/5, 12/5, 0, -12/5, 6/5)$, so scaling by 5/6 yields the new vector $p_3 = (5/6)(t^3 - (17/5)t) = (5/6)t^3 - (17/6)t$.

13. Suppose that A is invertible and that $\langle \mathbf{u}, \mathbf{v}\rangle = (A\mathbf{u})\cdot(A\mathbf{v})$ for $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Check each axiom in the definition on page 376, using the properties of the dot product.
    i. $\langle \mathbf{u}, \mathbf{v}\rangle = (A\mathbf{u})\cdot(A\mathbf{v}) = (A\mathbf{v})\cdot(A\mathbf{u}) = \langle \mathbf{v}, \mathbf{u}\rangle$
    ii. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w}\rangle = (A(\mathbf{u} + \mathbf{v}))\cdot(A\mathbf{w}) = (A\mathbf{u} + A\mathbf{v})\cdot(A\mathbf{w}) = (A\mathbf{u})\cdot(A\mathbf{w}) + (A\mathbf{v})\cdot(A\mathbf{w}) = \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle$
    iii. $\langle c\mathbf{u}, \mathbf{v}\rangle = (A(c\mathbf{u}))\cdot(A\mathbf{v}) = (c(A\mathbf{u}))\cdot(A\mathbf{v}) = c((A\mathbf{u})\cdot(A\mathbf{v})) = c\langle \mathbf{u}, \mathbf{v}\rangle$
    iv. $\langle \mathbf{u}, \mathbf{u}\rangle = (A\mathbf{u})\cdot(A\mathbf{u}) = \|A\mathbf{u}\|^2 \ge 0$, and this quantity is zero if and only if the vector $A\mathbf{u}$ is $\mathbf{0}$. But $A\mathbf{u} = \mathbf{0}$ if and only $\mathbf{u} = \mathbf{0}$ because A is invertible.

14. Suppose that T is a one-to-one linear transformation from a vector space V into $\mathbb{R}^n$ and that $\langle \mathbf{u}, \mathbf{v}\rangle = T(\mathbf{u})\cdot T(\mathbf{v})$ for $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Check each axiom in the definition on page 376, using the properties of the dot product and T. The linearity of T is used often in the following.
    i. $\langle \mathbf{u}, \mathbf{v}\rangle = T(\mathbf{u})\cdot T(\mathbf{v}) = T(\mathbf{v})\cdot T(\mathbf{u}) = \langle \mathbf{v}, \mathbf{u}\rangle$
    ii. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w}\rangle = T(\mathbf{u} + \mathbf{v})\cdot T(\mathbf{w}) = (T(\mathbf{u}) + T(\mathbf{v}))\cdot T(\mathbf{w}) = T(\mathbf{u})\cdot T(\mathbf{w}) + T(\mathbf{v})\cdot T(\mathbf{w}) = \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle$
    iii. $\langle c\mathbf{u}, \mathbf{v}\rangle = T(c\mathbf{u})\cdot T(\mathbf{v}) = (cT(\mathbf{u}))\cdot T(\mathbf{v}) = c(T(\mathbf{u})\cdot T(\mathbf{v})) = c\langle \mathbf{u}, \mathbf{v}\rangle$
    iv. $\langle \mathbf{u}, \mathbf{u}\rangle = T(\mathbf{u})\cdot T(\mathbf{u}) = \|T(\mathbf{u})\|^2 \ge 0$, and this quantity is zero if and only if $\mathbf{u} = \mathbf{0}$ since T is a one-to-one transformation.

15. Using Axioms 1 and 3, $\langle \mathbf{u}, c\mathbf{v}\rangle = \langle c\mathbf{v}, \mathbf{u}\rangle = c\langle \mathbf{v}, \mathbf{u}\rangle = c\langle \mathbf{u}, \mathbf{v}\rangle$.

16. Using Axioms 1, 2 and 3,

$$\begin{aligned}\|\mathbf{u}-\mathbf{v}\|^2 &= \langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{u}-\mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u}-\mathbf{v}\rangle \\ &= \langle \mathbf{u},\mathbf{u}\rangle - \langle \mathbf{u},\mathbf{v}\rangle - \langle \mathbf{v},\mathbf{u}\rangle + \langle \mathbf{v},\mathbf{v}\rangle = \langle \mathbf{u},\mathbf{u}\rangle - 2\langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{v},\mathbf{v}\rangle \\ &= \|\mathbf{u}\|^2 - 2\langle \mathbf{u},\mathbf{v}\rangle + \|\mathbf{v}\|^2\end{aligned}$$

Since $\{\mathbf{u}, \mathbf{v}\}$ is orthonormal, $\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2 = 1$ and $\langle \mathbf{u}, \mathbf{v}\rangle = 0$. So $\|\mathbf{u}-\mathbf{v}\|^2 = 2$.

17. Following the method in Exercise 16,

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{u}+\mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}+\mathbf{v}\rangle \\ &= \langle \mathbf{u},\mathbf{u}\rangle + \langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{v},\mathbf{u}\rangle + \langle \mathbf{v},\mathbf{v}\rangle = \langle \mathbf{u},\mathbf{u}\rangle + 2\langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{v},\mathbf{v}\rangle \\ &= \|\mathbf{u}\|^2 + 2\langle \mathbf{u},\mathbf{v}\rangle + \|\mathbf{v}\|^2\end{aligned}$$

Subtracting these results, one finds that $\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 = 4\langle \mathbf{u}, \mathbf{v}\rangle$, and dividing by 4 gives the desired identity.

18. In Exercises 16 and 17, it has been shown that $\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 - 2\langle \mathbf{u},\mathbf{v}\rangle + \|\mathbf{v}\|^2$ and $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\langle \mathbf{u},\mathbf{v}\rangle + \|\mathbf{v}\|^2$. Adding these two results gives $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$.

19. let $\mathbf{u} = \begin{bmatrix}\sqrt{a}\\ \sqrt{b}\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}\sqrt{b}\\ \sqrt{a}\end{bmatrix}$. Then $\|\mathbf{u}\|^2 = a+b$, $\|\mathbf{v}\|^2 = a+b$, and $\langle \mathbf{u}, \mathbf{v}\rangle = 2\sqrt{ab}$. Since a and b are nonnegative, $\|\mathbf{u}\| = \sqrt{a+b}$, $\|\mathbf{v}\| = \sqrt{a+b}$. Plugging these values into the Cauchy-Schwarz inequality gives

$$2\sqrt{ab} = |\langle \mathbf{u}, \mathbf{v}\rangle| \le \|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{a+b}\sqrt{a+b} = a+b$$

Dividing both sides of this equation by 2 gives the desired inequality.

20. The Cauchy-Schwarz inequality may be altered by dividing both sides of the inequality by 2 and then squaring both sides of the inequality. The result is

$$\left(\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{2}\right)^2 \le \frac{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}{4}$$

Now let $\mathbf{u} = \begin{bmatrix}a\\ b\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}1\\ 1\end{bmatrix}$. Then $\|\mathbf{u}\|^2 = a^2+b^2$, $\|\mathbf{v}\|^2 = 2$, and $\langle \mathbf{u}, \mathbf{v}\rangle = a+b$. Plugging these values into the inequality above yields the desired inequality.

21. The inner product is $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$. Let $f(t) = 1-3t^2$, $g(t) = t-t^3$. Then

$$\langle f, g\rangle = \int_0^1 (1-3t^2)(t-t^3)\,dt = \int_0^1 3t^5 - 4t^3 + t\,dt = 0$$

22. The inner product is $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$. Let $f(t) = 5t-3$, $g(t) = t^3-t^2$. Then

$$\langle f, g\rangle = \int_0^1 (5t-3)(t^3-t^2)dt = \int_0^1 5t^4 - 8t^3 + 3t^2 dt = 0$$

23. The inner product is $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$, so $\langle f, f\rangle = \int_0^1 (1-3t^2)^2 dt = \int_0^1 9t^4 - 6t^2 + 1\,dt = 4/5$, and $\|f\| = \sqrt{\langle f, f\rangle} = 2/\sqrt{5}$.

24. The inner product is $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$, so $\langle g, g\rangle = \int_0^1 (t^3-t^2)^2 dt = \int_0^1 t^6 - 2t^5 + t^4 dt = 1/105$, and $\|g\| = \sqrt{\langle g, g\rangle} = 1/\sqrt{105}$.

25. The inner product is $\langle f, g\rangle = \int_{-1}^1 f(t)g(t)dt$. Then 1 and t are orthogonal because $\langle 1, t\rangle = \int_{-1}^1 t\,dt = 0$. So 1 and t can be in an orthogonal basis for Span$\{1, t, t^2\}$. By the Gram-Schmidt process, the third basis element in the orthogonal basis can be

$$t^2 - \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle}1 - \frac{\langle t^2, t\rangle}{\langle t, t\rangle}t$$

Since $\langle t^2, 1\rangle = \int_{-1}^1 t^2 dt = 2/3$, $\langle 1, 1\rangle = \int_{-1}^1 1\,dt = 2$, and $\langle t^2, t\rangle = \int_{-1}^1 t^3 dt = 0$, the third basis element can be written as $t^2 - (1/3)$. This element can be scaled by 3, which gives the orthogonal basis as $\{1, t, 3t^2-1\}$.

26. The inner product is $\langle f, g\rangle = \int_{-2}^2 f(t)g(t)dt$. Then 1 and t are orthogonal because $\langle 1, t\rangle = \int_{-2}^2 t\,dt = 0$. So 1 and t can be in an orthogonal basis for Span$\{1, t, t^2\}$. By the Gram-Schmidt process, the third basis element in the orthogonal basis can be

$$t^2 - \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle}1 - \frac{\langle t^2, t\rangle}{\langle t, t\rangle}t$$

Since $\langle t^2, 1\rangle = \int_{-2}^2 t^2 dt = 16/3$, $\langle 1, 1\rangle = \int_{-2}^2 1\,dt = 4$, and $\langle t^2, t\rangle = \int_{-2}^2 t^3 dt = 0$, the third basis element can be written as $t^2 - (4/3)$. This element can be scaled by 3, which gives the orthogonal basis as $\{1, t, 3t^2-4\}$.

27. [M] The new orthogonal polynomials are multiples of $-17t + 5t^3$ and $72 - 155t^2 + 35t^4$. These polynomials may be scaled so that their values at –2, –1, 0, 1, and 2 are small integers.

28. [M] The orthogonal basis is $f_0(t) = 1$, $f_1(t) = \cos t$, $f_2(t) = \cos^2 t - (1/2) = (1/2)\cos 2t$, and $f_3(t) = \cos^3 t - (3/4)\cos t = (1/4)\cos 3t$.