6.4 SOLUTIONS

Notes: The QR factorization encapsulates the essential outcome of the Gram-Schmidt process, just as the LU factorization describes the result of a row reduction process. For practical use of linear algebra, the factorizations are more important than the algorithms that produce them. In fact, the Gram-Schmidt process is *not* the appropriate way to compute the QR factorization. For that reason, one should consider deemphasizing the hand calculation of the Gram-Schmidt process, even though it provides easy exam questions.

The Gram-Schmidt process is used in Sections 6.7 and 6.8, in connection with various sets of orthogonal polynomials. The process is mentioned in Sections 7.1 and 7.4, but the one-dimensional projection constructed in Section 6.2 will suffice. The QR factorization is used in an optional subsection of Section 6.5, and it is needed in Supplementary Exercise 7 of Chapter 7 to produce the Cholesky factorization of a positive definite matrix.

1. Set $\mathbf{v}_1 = \mathbf{x}_1$ and compute that $\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - 3\mathbf{v}_1 = \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}$. Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} \right\}$.

2. Set $\mathbf{v}_1 = \mathbf{x}_1$ and compute that $\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - \frac{1}{2}\mathbf{v}_1 = \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}$. Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix} \right\}$.

3. Set $\mathbf{v}_1 = \mathbf{x}_1$ and compute that $\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - \frac{1}{2}\mathbf{v}_1 = \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix}$. Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix} \right\}$.

4. Set $\mathbf{v}_1 = \mathbf{x}_1$ and compute that $\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - (-2)\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$. Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} \right\}$.

5. Set $\mathbf{v}_1 = \mathbf{x}_1$ and compute that $\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - 2\mathbf{v}_1 = \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix}$. Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix} \right\}$.

6. Set $\mathbf{v}_1 = \mathbf{x}_1$ and compute that $\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - (-3)\mathbf{v}_1 = \begin{bmatrix} 4 \\ 6 \\ -3 \\ 0 \end{bmatrix}$. Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ -3 \\ 0 \end{bmatrix} \right\}$.

7. Since $\|\mathbf{v}_1\| = \sqrt{30}$ and $\|\mathbf{v}_2\| = \sqrt{27/2} = 3\sqrt{6}/2$, an orthonormal basis for W is

$$\left\{ \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} \right\} = \left\{ \begin{bmatrix} 2/\sqrt{30} \\ -5/\sqrt{30} \\ 1/\sqrt{30} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}.$$

8. Since $\|\mathbf{v}_1\| = \sqrt{50}$ and $\|\mathbf{v}_2\| = \sqrt{54} = 3\sqrt{6}$, an orthonormal basis for W is

$$\left\{ \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} \right\} = \left\{ \begin{bmatrix} 3/\sqrt{50} \\ -4/\sqrt{50} \\ 5/\sqrt{50} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}.$$

9. Call the columns of the matrix $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ and perform the Gram-Schmidt process on these vectors:

$$\mathbf{v}_1 = \mathbf{x}_1$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - (-2)\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \mathbf{x}_3 - \frac{3}{2}\mathbf{v}_1 - \left(-\frac{1}{2}\right)\mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$.

10. Call the columns of the matrix $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ and perform the Gram-Schmidt process on these vectors:

$\mathbf{v}_1 = \mathbf{x}_1$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - (-3)\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \mathbf{x}_3 - \frac{1}{2}\mathbf{v}_1 - \frac{5}{2}\mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \right\}$.

11. Call the columns of the matrix $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ and perform the Gram-Schmidt process on these vectors:

$\mathbf{v}_1 = \mathbf{x}_1$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - (-1)\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \mathbf{x}_3 - 4\mathbf{v}_1 - \left(-\frac{1}{3}\right)\mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix}$$

Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix} \right\}$.

12. Call the columns of the matrix $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ and perform the Gram-Schmidt process on these vectors:

$\mathbf{v}_1 = \mathbf{x}_1$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - 4\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \mathbf{x}_3 - \frac{7}{2}\mathbf{v}_1 - \frac{3}{2}\mathbf{v}_2 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix}$$

Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix} \right\}$.

13. Since A and Q are given,

$$R = Q^T A = \begin{bmatrix} 5/6 & 1/6 & -3/6 & 1/6 \\ -1/6 & 5/6 & 1/6 & 3/6 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix}$$

14. Since A and Q are given,

$$R = Q^T A = \begin{bmatrix} -2/7 & 5/7 & 2/7 & 4/7 \\ 5/7 & 2/7 & -4/7 & 2/7 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix}$$

15. The columns of Q will be normalized versions of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ found in Exercise 11. Thus

$$Q=\begin{bmatrix} 1/\sqrt{5} & 1/2 & 1/2 \\ -1/\sqrt{5} & 0 & 0 \\ -1/\sqrt{5} & 1/2 & 1/2 \\ 1/\sqrt{5} & -1/2 & 1/2 \\ 1/\sqrt{5} & 1/2 & -1/2 \end{bmatrix}, R=Q^TA=\begin{bmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

16. The columns of Q will be normalized versions of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ found in Exercise 12. Thus

$$Q=\begin{bmatrix} 1/2 & -1/(2\sqrt{2}) & 1/2 \\ -1/2 & 1/(2\sqrt{2}) & 1/2 \\ 0 & 1/\sqrt{2} & 0 \\ 1/2 & 1/(2\sqrt{2}) & -1/2 \\ 1/2 & 1/(2\sqrt{2}) & 1/2 \end{bmatrix}, R=Q^TA=\begin{bmatrix} 2 & 8 & 7 \\ 0 & 2\sqrt{2} & 3\sqrt{2} \\ 0 & 0 & 6 \end{bmatrix}$$

17. **a.** False. Scaling was used in Example 2, but the scale factor was nonzero.

b. True. See (1) in the statement of Theorem 11.

c. True. See the solution of Example 4.

18. **a.** False. The three orthogonal vectors must be *nonzero* to be a basis for a three-dimensional subspace. (This was the case in Step 3 of the solution of Example 2.)

b. True. If $\mathbf{x}$ is not in a subspace w, then $\mathbf{x}$ cannot equal $\text{proj}_W\mathbf{x}$, because $\text{proj}_W\mathbf{x}$ is in W. This idea was used for $\mathbf{v}_{k+1}$ in the proof of Theorem 11.

c. True. See Theorem 12.

19. Suppose that $\mathbf{x}$ satisfies $R\mathbf{x}=\mathbf{0}$; then $QR\mathbf{x}=Q\mathbf{0}=\mathbf{0}$, and $A\mathbf{x}=\mathbf{0}$. Since the columns of A are linearly independent, $\mathbf{x}$ must be $\mathbf{0}$. This fact, in turn, shows that the columns of R are linearly indepedent. Since R is square, it is invertible by the Invertible Matrix Theorem.

20. If $\mathbf{y}$ is in Col$A$, then $\mathbf{y}=A\mathbf{x}$ for some $\mathbf{x}$. Then $\mathbf{y}=QR\mathbf{x}=Q(R\mathbf{x})$, which shows that $\mathbf{y}$ is a linear combination of the columns of Q using the entries in $R\mathbf{x}$ as weights. Conversly, suppose that $\mathbf{y}=Q\mathbf{x}$ for some $\mathbf{x}$. Since R is invertible, the equation $A=QR$ implies that $Q=AR^{-1}$. So $\mathbf{y}=AR^{-1}\mathbf{x}=A(R^{-1}\mathbf{x})$, which shows that $\mathbf{y}$ is in Col A.

21. Denote the columns of Q by $\{\mathbf{q}_1,\dots,\mathbf{q}_n\}$. Note that $n\le m$, because A is $m\times n$ and has linearly independent columns. The columns of Q can be extended to an orthonormal basis for $\mathbb{R}^m$ as follows. Let $\mathbf{f}_1$ be the first vector in the standard basis for $\mathbb{R}^m$ that is *not* in $W_n=\text{Span}\{\mathbf{q}_1,\dots,\mathbf{q}_n\}$, let $\mathbf{u}_1=\mathbf{f}_1-\text{proj}_{W_n}\mathbf{f}_1$, and let $\mathbf{q}_{n+1}=\mathbf{u}_1/\|\mathbf{u}_1\|$. Then $\{\mathbf{q}_1,\dots,\mathbf{q}_n,\mathbf{q}_{n+1}\}$ is an orthonormal basis for $W_{n+1}=\text{Span}\{\mathbf{q}_1,\dots,\mathbf{q}_n,\mathbf{q}_{n+1}\}$. Next let $\mathbf{f}_2$ be the first vector in the standard basis for $\mathbb{R}^m$ that is *not* in W_{n+1}, let $\mathbf{u}_2=\mathbf{f}_2-\text{proj}_{W_{n+1}}\mathbf{f}_2$, and let $\mathbf{q}_{n+2}=\mathbf{u}_2/\|\mathbf{u}_2\|$. Then $\{\mathbf{q}_1,\dots,\mathbf{q}_n,\mathbf{q}_{n+1},\mathbf{q}_{n+2}\}$ is an orthogonal basis for $W_{n+2}=\text{Span}\{\mathbf{q}_1,\dots,\mathbf{q}_n,\mathbf{q}_{n+1},\mathbf{q}_{n+2}\}$. This process will continue until $m-n$ vectors have been added to the original n vectors, and $\{\mathbf{q}_1,\dots,\mathbf{q}_n,\mathbf{q}_{n+1},\dots,\mathbf{q}_m\}$ is an orthonormal basis for $\mathbb{R}^m$.

Let $Q_0=[\mathbf{q}_{n+1} \ \ \dots \ \ \mathbf{q}_m]$ and $Q_1=[Q \ \ Q_0]$. Then, using partitioned matrix multiplication, $Q_1\begin{bmatrix} R \\ O\end{bmatrix}=QR=A$.

22. We may assume that $\{\mathbf{u}_1,\dots,\mathbf{u}_p\}$ is an orthonormal basis for W, by normalizing the vectors in the original basis given for W, if necessary. Let U be the matrix whose columns are $\mathbf{u}_1,\dots,\mathbf{u}_p$. Then, by Theorem 10 in Section 6.3, $T(\mathbf{x})=\text{proj}_W\mathbf{x}=(UU^T)\mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^n$. Thus T is a matrix transformation and hence is a linear transformation, as was shown in Section 1.8.

23. Given $A=QR$, partition $A=[A_1 \ \ A_2]$, where A_1 has p columns. Partition Q as $Q=[Q_1 \ \ Q_2]$ where Q_1 has p columns, and partition R as $R=\begin{bmatrix} R_{11} & R_{12} \\ O & R_{22}\end{bmatrix}$, where R_{11} is a $p\times p$ matrix. Then

$$A=[A_1 \ \ A_2]=QR=[Q_1 \ \ Q_2]\begin{bmatrix} R_{11} & R_{12} \\ O & R_{22}\end{bmatrix}=[Q_1R_{11} \ \ Q_1R_{12}+Q_2R_{22}]$$

Thus $A_1=Q_1R_{11}$. The matrix Q_1 has orthonormal columns because its columns come from Q. The matrix R_{11} is square and upper triangular due to its position within the upper triangular matrix R. The diagonal entries of R_{11} are positive because they are diagonal entries of R. Thus Q_1R_{11} is a QR factorization of A_1.

24. **[M]** Call the columns of the matrix $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ and perform the Gram-Schmidt process on these vectors:

$$\mathbf{v}_1=\mathbf{x}_1$$

$$\mathbf{v}_2=\mathbf{x}_2-\frac{\mathbf{x}_2\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1=\mathbf{x}_2-(-1)\mathbf{v}_1=\begin{bmatrix}3\\3\\-3\\0\\3\end{bmatrix}$$

$$\mathbf{v}_3=\mathbf{x}_3-\frac{\mathbf{x}_3\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1-\frac{\mathbf{x}_3\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2=\mathbf{x}_3-\left(-\frac{1}{2}\right)\mathbf{v}_1-\left(-\frac{4}{3}\right)\mathbf{v}_2=\begin{bmatrix}6\\0\\6\\6\\0\end{bmatrix}$$

$$\mathbf{v}_4=\mathbf{x}_4-\frac{\mathbf{x}_4\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1-\frac{\mathbf{x}_4\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2-\frac{\mathbf{x}_4\cdot\mathbf{v}_3}{\mathbf{v}_3\cdot\mathbf{v}_3}\mathbf{v}_3=\mathbf{x}_4-\frac{1}{2}\mathbf{v}_1-(-1)\mathbf{v}_2-\left(-\frac{1}{2}\right)\mathbf{v}_3=\begin{bmatrix}0\\5\\0\\0\\-5\end{bmatrix}$$