1. a. True. If A is invertible and if $A\mathbf{x} = 1\cdot\mathbf{x}$ for some nonzero $\mathbf{x}$, then left-multiply by A^{-1} to obtain $\mathbf{x} = A^{-1}\mathbf{x}$, which may be rewritten as $A^{-1}\mathbf{x} = 1\cdot\mathbf{x}$. Since $\mathbf{x}$ is nonzero, this shows 1 is an eigenvalue of A^{-1}.

b. False. If A is row equivalent to the identity matrix, then A is invertible. The matrix in Example 4 of Section 5.3 shows that an invertible matrix need not be diagonalizable. Also, see Exercise 31 in Section 5.3.

c. True. If A contains a row or column of zeros, then A is not row equivalent to the identity matrix and thus is not invertible. By the Invertible Matrix Theorem (as stated in Section 5.2), 0 is an eigenvalue of A.

d. False. Consider a diagonal matrix D whose eigenvalues are 1 and 3, that is, its diagonal entries are 1 and 3. Then D^2 is a diagonal matrix whose eigenvalues (diagonal entries) are 1 and 9. In general, the eigenvalues of A^2 are the *squares* of the eigenvalues of A.

e. True. Suppose a nonzero vector $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$, then

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$$

This shows that $\mathbf{x}$ is also an eigenvector for A^2

f. True. Suppose a nonzero vector $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$, then left-multiply by A^{-1} to obtain $\mathbf{x} = A^{-1}(\lambda\mathbf{x}) = \lambda A^{-1}\mathbf{x}$. Since A is invertible, the eigenvalue λ is not zero. So $\lambda^{-1}\mathbf{x} = A^{-1}\mathbf{x}$, which shows that $\mathbf{x}$ is also an eigenvector of A^{-1}.

g. False. Zero is an eigenvalue of each singular square matrix.

h. True. By definition, an eigenvector must be nonzero.

i. False. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ then $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors of A for the eigenvalue 2, and they are linearly independent.

j. True. This follows from Theorem 4 in Section 5.2

k. False. Let A be the 3×3 matrix in Example 3 of Section 5.3. Then A is similar to a diagonal matrix D. The eigenvectors of D are the columns of I_3, but the eigenvectors of A are entirely different.

l. False. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors of A, but $\mathbf{e}_1 + \mathbf{e}_2$ is not. (Actually, it can be shown that if two eigenvectors of A correspond to distinct eigenvalues, then their sum cannot be an eigenvector.)

m. False. *All* the diagonal entries of an upper triangular matrix are the eigenvalues of the matrix (Theorem 1 in Section 5.1). A diagonal entry may be zero.

n. True. Matrices A and A^T have the same characteristic polynomial, because $\det(A^T - \lambda I) = \det(A - \lambda I)^T = \det(A - \lambda I)$, by the determinant transpose property.

o. False. Counterexample: Let A be the 5×5 identity matrix.

p. True. For example, let A be the matrix that rotates vectors through $\pi/2$ radians about the origin. Then $A\mathbf{x}$ is not a multiple of $\mathbf{x}$ when $\mathbf{x}$ is nonzero.

q. False. If A is a diagonal matrix with 0 on the diagonal, then the columns of A are not linearly independent.

r. True. If $A\mathbf{x} = \lambda_1\mathbf{x}$ and $A\mathbf{x} = \lambda_2\mathbf{x}$, then $\lambda_1\mathbf{x} = \lambda_2\mathbf{x}$ and $(\lambda_1 - \lambda_2)\mathbf{x} = \mathbf{0}$. If $\mathbf{x} \neq \mathbf{0}$, then λ_1 must equal λ_2.

s. False. Let A be a singular matrix that is diagonalizable. (For instance, let A be a diagonal matrix with 0 on the diagonal.) Then, by Theorem 8 in Section 5.4, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is represented by a diagonal matrix relative to a coordinate system determined by eigenvectors of A.

t. True. By definition of matrix multiplication,

$$A = AI = A[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n] = [A\mathbf{e}_1 \quad A\mathbf{e}_2 \quad \cdots \quad A\mathbf{e}_n]$$

If $A\mathbf{e}_j = d_j\mathbf{e}_j$ for $j = 1,\dots,n$, then A is a diagonal matrix with diagonal entries $d_1,\dots,d_n$.

u. True. If $B = PDP^{-1}$, where D is a diagonal matrix, and if $A = QBQ^{-1}$, then $A = Q(PDP^{-1})Q^{-1} = (QP)D(QP)^{-1}$, which shows that A is diagonalizable.

v. True. Since B is invertible, AB is similar to $B(AB)B^{-1}$, which equals BA.

w. False. Having n linearly independent eigenvectors makes an $n\times n$ matrix diagonalizable (by the Diagonalization Theorem 5 in Section 5.3), but not necessarily invertible. One of the eigenvalues of the matrix could be zero.

x. True. If A is diagonalizable, then by the Diagonalization Theorem, A has n linearly independent eigenvectors $\mathbf{v}_1,\dots,\mathbf{v}_n$ in $\mathbf{R}^n$. By the Basis Theorem, $\{\mathbf{v}_1,\dots,\mathbf{v}_n\}$ spans $\mathbf{R}^n$. This means that each vector in $\mathbf{R}^n$ can be written as a linear combination of $\mathbf{v}_1,\dots,\mathbf{v}_n$.

2. Suppose $B\mathbf{x} \neq \mathbf{0}$ and $AB\mathbf{x} = \lambda\mathbf{x}$ for some λ. Then $A(B\mathbf{x}) = \lambda\mathbf{x}$. Left-multiply each side by B and obtain $BA(B\mathbf{x}) = B(\lambda\mathbf{x}) = \lambda(B\mathbf{x})$. This equation says that $B\mathbf{x}$ is an eigenvector of BA, because $B\mathbf{x} \neq \mathbf{0}$.

3. a. Suppose $A\mathbf{x} = \lambda\mathbf{x}$, with $\mathbf{x} \neq \mathbf{0}$. Then $(5I - A)\mathbf{x} = 5\mathbf{x} - A\mathbf{x} = 5\mathbf{x} - \lambda\mathbf{x} = (5 - \lambda)\mathbf{x}$. The eigenvalue is $5 - \lambda$.

b. $(5I - 3A + A^2)\mathbf{x} = 5\mathbf{x} - 3A\mathbf{x} + A(A\mathbf{x}) = 5\mathbf{x} - 3(\lambda\mathbf{x}) + \lambda^2\mathbf{x} = (5 - 3\lambda + \lambda^2)\mathbf{x}$. The eigenvalue is $5 - 3\lambda + \lambda^2$.

4. Assume that $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x}$. The desired statement is true for $m = 1$, by the assumption about λ. Suppose that for some $k \geq 1$, the statement holds when $m = k$. That is, suppose that $A^k\mathbf{x} = \lambda^k\mathbf{x}$. Then $A^{k+1}\mathbf{x} = A(A^k\mathbf{x}) = A(\lambda^k\mathbf{x})$ by the induction hypothesis. Continuing, $A^{k+1}\mathbf{x} = \lambda^k A\mathbf{x} = \lambda^{k+1}\mathbf{x}$, because $\mathbf{x}$ is an eigenvector of A corresponding to A. Since $\mathbf{x}$ is nonzero, this equation shows that λ^{k+1} is an eigenvalue of A^{k+1}, with corresponding eigenvector $\mathbf{x}$. Thus the desired statement is true when $m = k + 1$. By the principle of induction, the statement is true for each positive integer m.

5. Suppose $A\mathbf{x} = \lambda\mathbf{x}$, with $\mathbf{x} \neq \mathbf{0}$. Then

$$\begin{aligned} p(A)\mathbf{x} &= (c_0 I + c_1 A + c_2 A^2 + \ldots + c_n A^n)\mathbf{x} \\ &= c_0\mathbf{x} + c_1 A\mathbf{x} + c_2 A^2\mathbf{x} + \ldots + c_n A^n\mathbf{x} \\ &= c_0\mathbf{x} + c_1\lambda\mathbf{x} + c_2\lambda^2\mathbf{x} + \ldots + c_n\lambda^n\mathbf{x} = p(\lambda)\mathbf{x} \end{aligned}$$

So $p(\lambda)$ is an eigenvalue of $p(A)$.

6. a. If $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$, and

$$\begin{aligned} B = 5I - 3A + A^2 &= 5PIP^{-1} - 3PDP^{-1} + PD^2P^{-1} \\ &= P(5I - 3D + D^2)P^{-1} \end{aligned}$$

b. $p(A) = c_0 I + c_1 PDP^{-1} + c_2 PD^2 P^{-1} + \cdots + c_n PD^n P^{-1}$
$= P(c_0 I + c_1 D + c_2 D^2 + \cdots + c_n D^n)P^{-1}$
$= Pp(D)P^{-1}$

This shows that $p(A)$ is diagonalizable, because $p(D)$ is a linear combination of diagonal matrices and hence is diagonal. In fact, because *D* is diagonal, it is easy to see that

$$p(D) = \begin{bmatrix} p(2) & 0 \\ 0 & p(7) \end{bmatrix}$$

7. If $A = PDP^{-1}$, then $p(A) = Pp(D)P^{-1}$, as shown in Exercise 6. If the (j, j) entry in *D* is λ, then the (j, j) entry in D^k is λ^k, and so the (j, j) entry in $p(D)$ is $p(\lambda)$. If *p* is the characteristic polynomial of *A*, then $p(\lambda) = 0$ for each diagonal entry of *D*, because these entries in *D* are the eigenvalues of *A*. Thus $p(D)$ is the zero matrix. Thus $p(A) = P \cdot 0 \cdot P^{-1} = 0$.

8. a. If λ is an eigenvalue of an $n \times n$ diagonalizable matrix *A*, then $A = PDP^{-1}$ for an invertible matrix *P* and an $n \times n$ diagonal matrix *D* whose diagonal entries are the eigenvalues of *A*. If the multiplicity of λ is *n*, then λ must appear in every diagonal entry of *D*. That is, $D = \lambda I$. In this case, $A = P(\lambda I)P^{-1} = \lambda PIP^{-1} = \lambda PP^{-1} = \lambda I$.

b. Since the matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is triangular, its eigenvalues are on the diagonal. Thus 3 is an eigenvalue with multiplicity 2. If the 2×2 matrix *A* were diagonalizable, then *A* would be 3*I*, by part (a). This is not the case, so *A* is not diagonalizable.

9. If $I - A$ were not invertible, then the equation $(I - A)\mathbf{x} = \mathbf{0}$. would have a nontrivial solution **x**. Then $\mathbf{x} - A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = 1 \cdot \mathbf{x}$, which shows that *A* would have 1 as an eigenvalue. This cannot happen if all the eigenvalues are less than 1 in magnitude. So $I - A$ must be invertible.

10. To show that A^k tends to the zero matrix, it suffices to show that each column of A^k can be made as close to the zero vector as desired by taking *k* sufficiently large. The *j*th column of *A* is $A\mathbf{e}_j$, where $\mathbf{e}_j$ is the *j*th column of the identity matrix. Since *A* is diagonalizable, there is a basis for $\mathbb{R}^n$ consisting of eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, corresponding to eigenvalues $\lambda_1, \ldots, \lambda_n$. So there exist scalars $c_1, \ldots, c_n$, such that

$\mathbf{e}_j = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ (an eigenvector decomposition of $\mathbf{e}_j$)

Then, for $k = 1, 2, \ldots$,

$$A^k\mathbf{e}_j = c_1(\lambda_1)^k\mathbf{v}_1 + \cdots + c_n(\lambda_n)^k\mathbf{v}_n \qquad (*)$$

If the eigenvalues are all less than 1 in absolute value, then their *k*th powers all tend to zero. So (*) shows that $A^k\mathbf{e}_j$ tends to the zero vector, as desired.

11. a. Take **x** in *H*. Then $\mathbf{x} = c\mathbf{u}$ for some scalar *c*. So $A\mathbf{x} = A(c\mathbf{u}) = c(A\mathbf{u}) = c(\lambda\mathbf{u}) = (c\lambda)\mathbf{u}$, which shows that $A\mathbf{x}$ is in *H*.

b. Let **x** be a nonzero vector in *K*. Since *K* is one-dimensional, *K* must be the set of all scalar multiples of **x**. If *K* is invariant under *A*, then $A\mathbf{x}$ is in *K* and hence $A\mathbf{x}$ is a multiple of **x**. Thus **x** is an eigenvector of *A*.

12. Let *U* and *V* be echelon forms of *A* and *B*, obtained with *r* and *s* row interchanges, respectively, and no scaling. Then $\det A = (-1)^r \det U$ and $\det B = (-1)^s \det V$

Using first the row operations that reduce *A* to *U*, we can reduce *G* to a matrix of the form $G' = \begin{bmatrix} U & Y \\ 0 & B \end{bmatrix}$. Then, using the row operations that reduce *B* to *V*, we can further reduce G' to $G'' = \begin{bmatrix} U & Y \\ 0 & V \end{bmatrix}$. There will be $r + s$ row interchanges, and so

$\det G = \det \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} = (-1)^{r+s} \det \begin{bmatrix} U & Y \\ 0 & V \end{bmatrix}$ Since $\begin{bmatrix} U & Y \\ 0 & V \end{bmatrix}$ is upper triangular, its determinant equals the product of the diagonal entries,
and since *U* and *V* are upper triangular, this product also equals (det *U*) (det *V*). Thus

$\det G = (-1)^{r+s}(\det U)(\det V) = (\det A)(\det B)$

For any scalar λ, the matrix $G - \lambda I$ has the same partitioned form as *G*, with $A - \lambda I$ and $B - \lambda I$ as its diagonal blocks. (Here *I* represents various identity matrices of appropriate sizes.) Hence the result about det *G* shows that $\det(G - \lambda I) = \det(A - \lambda I) \cdot \det(B - \lambda I)$

13. By Exercise 12, the eigenvalues of *A* are the eigenvalues of the matrix $[3]$ together with the eigenvalues of $\begin{bmatrix} 5 & -2 \\ -4 & 3 \end{bmatrix}$. The only eigenvalue of $[3]$ is 3, while the eigenvalues of $\begin{bmatrix} 5 & -2 \\ -4 & 3 \end{bmatrix}$ are 1 and 7. Thus the eigenvalues of *A* are 1, 3, and 7.

14. By Exercise 12, the eigenvalues of *A* are the eigenvalues of the matrix $\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$ together with the eigenvalues of $\begin{bmatrix} -7 & -4 \\ 3 & 1 \end{bmatrix}$. The eigenvalues of $\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$ are -1 and 6, while the eigenvalues of $\begin{bmatrix} -7 & -4 \\ 3 & 1 \end{bmatrix}$ are -5 and -1. Thus the eigenvalues of *A* are $-1, -5$, and 6, and the eigenvalue -1 has multiplicity 2.

15. Replace *a* by $a - \lambda$ in the determinant formula from Exercise 16 in Chapter 3 Supplementary Exercises.

$\det(A - \lambda I) = (a - b - \lambda)^{n-1}[a - \lambda + (n-1)b]$

This determinant is zero only if $a - b - \lambda = 0$ or $a - \lambda + (n-1)b = 0$. Thus λ is an eigenvalue of *A* if and only if $\lambda = a - b$ or $\lambda = a + (n-1)$. From the formula for $\det(A - \lambda I)$ above, the algebraic multiplicity is $n - 1$ for $a - b$ and 1 for $a + (n-1)b$.

16. The 3×3 matrix has eigenvalues $1 - 2$ and $1 + (2)(2)$, that is, -1 and 5. The eigenvalues of the 5×5 matrix are $7 - 3$ and $7 + (4)(3)$, that is 4 and 19.

17. Note that $\det(A-\lambda I)=(a_{11}-\lambda)(a_{22}-\lambda)-a_{12}a_{21}=\lambda^2-(a_{11}+a_{22})\lambda+(a_{11}a_{22}-a_{12}a_{21})$ $=\lambda^2-(\text{tr }A)\lambda+\det A,$ and use the quadratic formula to solve the characteristic equation:

$$\lambda=\frac{\text{tr }A\pm\sqrt{(\text{tr }A)^2-4\det A}}{2}$$

The eigenvalues are both real if and only if the discriminant is nonnegative, that is, $(\text{tr }A)^2-4\det A\geq 0.$ This inequality simplifies to $(\text{tr }A)^2\geq 4\det A$ and $\left(\frac{trA}{2}\right)^2\geq\det A.$

18. The eigenvalues of A are 1 and .6. Use this to factor A and A^k.

$$A=\begin{bmatrix}-1&-3\\2&2\end{bmatrix}\begin{bmatrix}1&0\\0&.6\end{bmatrix}\cdot\frac{1}{4}\begin{bmatrix}2&3\\-2&-1\end{bmatrix}$$

$$A^k=\begin{bmatrix}-1&-3\\2&2\end{bmatrix}\begin{bmatrix}1^k&0\\0&.6^k\end{bmatrix}\cdot\frac{1}{4}\begin{bmatrix}2&3\\-2&-1\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}-1&-3\\2&2\end{bmatrix}\begin{bmatrix}2&3\\-2\cdot(.6)^k&-(.6)^k\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}-2+6(.6)^k&-3+3(.6)^k\\4-4(.6)^k&6-2(.6)^k\end{bmatrix}$$

$$\to\frac{1}{4}\begin{bmatrix}-2&-3\\4&6\end{bmatrix}\text{ as }k\to\infty$$

19. $C_p=\begin{bmatrix}0&1\\-6&5\end{bmatrix};\det(C_p-\lambda I)=6-5\lambda+\lambda^2=p(\lambda)$

20. $C_p=\begin{bmatrix}0&1&0\\0&0&1\\24&-26&9\end{bmatrix};$

$\det(C_p-\lambda I)=24-26\lambda+9\lambda^2-\lambda^3=p(\lambda)$

21. If p is a polynomial of order 2, then a calculation such as in Exercise 19 shows that the characteristic polynomial of C_p is $p(\lambda)=(-1)^2p(\lambda),$ so the result is true for $n=2.$ Suppose the result is true for $n=k$ for some $k\geq 2,$ and consider a polynomial p of degree $k+1.$ Then expanding $\det(C_p-\lambda I)$ by cofactors down the first column, the determinant of $C_p-\lambda I$ equals

$$(-\lambda)\det\begin{bmatrix}-\lambda&1&\cdots&0\\\vdots&&&\vdots\\0&&&1\\-a_1&-a_2&\cdots&-a_k-\lambda\end{bmatrix}+(-1)^{k+1}a_0$$

The $k\times k$ matrix shown is $C_q-\lambda I,$ where $q(t)=a_1+a_2t+\cdots+a_kt^{k-1}+t^k.$ By the induction assumption, the determinant of $C_q-\lambda I$ is $(-1)^kq(\lambda).$ Thus

$$\begin{aligned}\det(C_p-\lambda I)&=(-1)^{k+1}a_0+(-\lambda)(-1)^kq(\lambda)\\&=(-1)^{k+1}[a_0+\lambda(a_1+\cdots+a_k\lambda^{k-1}+\lambda^k)]\\&=(-1)^{k+1}p(\lambda)\end{aligned}$$

So the formula holds for $n=k+1$ when it holds for $n=k.$ By the principle of induction, the formula for $\det(C_p-\lambda I)$ is true for all $n\geq 2.$

22. **a.** $C_p=\begin{bmatrix}0&1&0\\0&0&1\\-a_0&-a_1&-a_2\end{bmatrix}$

b. Since λ is a zero of $p,$ $a_0+a_1\lambda+a_2\lambda^2+\lambda^3=0$ and $-a_0-a_1\lambda-a_2\lambda^2=\lambda^3.$ Thus

$$C_p\begin{bmatrix}1\\\lambda\\\lambda^2\end{bmatrix}=\begin{bmatrix}\lambda\\\lambda^2\\-a_0-a_1\lambda-a_2\lambda^2\end{bmatrix}=\begin{bmatrix}\lambda\\\lambda^2\\\lambda^3\end{bmatrix}$$

That is, $C_p(1,\lambda,\lambda^2)=\lambda(1,\lambda,\lambda^2),$ which shows that $(1,\lambda,\lambda^2)$ is an eigenvector of C_p corresponding to the eigenvalue λ.

23. From Exercise 22, the columns of the Vandermonde matrix V are eigenvectors of C_p, corresponding to the eigenvalues $\lambda_1,\lambda_2,\lambda_3$ (the roots of the polynomial p). Since these eigenvalues are distinct, the eigenvectors from a linearly independent set, by Theorem 2 in Section 5.1. Thus V has linearly independent columns and hence is invertible, by the Invertible Matrix Theorem. Finally, since the columns of V are eigenvectors of C_p, the Diagonalization Theorem (Theorem 5 in Section 5.3) shows that $V^{-1}C_pV$ is diagonal.

24. **[M]** The MATLAB command `roots(p)` requires as input a row vector p whose entries are the coefficients of a polynomial, with the highest order coefficient listed first. MATLAB constructs a companion matrix C_p whose characteristic polynomial is p, so the roots of p are the eigenvalues of C_p. The numerical values of the eigenvalues (roots) are found by the same QR algorithm used by the command `eig(A)`.

25. **[M]** The MATLAB command `[P D]=eig(A)` produces a matrix P, whose condition number is 1.6×10^8, and a diagonal matrix D, whose entries are *almost* 2, 2, 1. However, the exact eigenvalues of A are 2, 2, 1, and A is not diagonalizable.

26. **[M]** This matrix may cause the same sort of trouble as the matrix in Exercise 25. A matrix program that computes eigenvalues by an interative process may indicate that A has four distinct eigenvalues, all close to zero. However, the only eigenvalue is 0, with multiplicity 4, because $A^4=0$.