where c_1 and c_2 now are real numbers. To satisfy the initial condition $\mathbf{x}(0)=\begin{bmatrix}0\\12\end{bmatrix}$, we solve

$c_1\begin{bmatrix}-1\\1\end{bmatrix}+c_2\begin{bmatrix}-2\\0\end{bmatrix}=\begin{bmatrix}0\\12\end{bmatrix}$ to get $c_1=12, c_2=-6$. We now have

$$\begin{bmatrix}i_L(t)\\v_C(t)\end{bmatrix}=\mathbf{x}(t)=12\begin{bmatrix}-\cos .8t+2\sin .8t\\\cos .8t\end{bmatrix}e^{-.4t}-6\begin{bmatrix}-\sin .8t-2\cos .8t\\\sin .8t\end{bmatrix}e^{-.4t}=\begin{bmatrix}30\sin .8t\\12\cos .8t-6\sin .8t\end{bmatrix}e^{-.4t}$$

5.8 SOLUTIONS

1. The vectors in the given sequence approach an eigenvector $\mathbf{v}_1$. The last vector in the sequence, $\mathbf{x}_4=\begin{bmatrix}1\\.3326\end{bmatrix}$, is probably the best estimate for $\mathbf{v}_1$. To compute an estimate for λ_1, examine $A\mathbf{x}_4=\begin{bmatrix}4.9978\\1.6652\end{bmatrix}$. This vector is approximately $\lambda_1\mathbf{v}_1$. From the first entry in this vector, an estimate of λ_1 is 4.9978.

2. The vectors in the given sequence approach an eigenvector $\mathbf{v}_1$. The last vector in the sequence, $\mathbf{x}_4=\begin{bmatrix}-.2520\\1\end{bmatrix}$, is probably the best estimate for $\mathbf{v}_1$. To compute an estimate for λ_1, examine $A\mathbf{x}_4=\begin{bmatrix}-1.2536\\5.0064\end{bmatrix}$. This vector is approximately $\lambda_1\mathbf{v}_1$. From the second entry in this vector, an estimate of λ_1 is 5.0064.

3. The vectors in the given sequence approach an eigenvector $\mathbf{v}_1$. The last vector in the sequence, $\mathbf{x}_4=\begin{bmatrix}.5188\\1\end{bmatrix}$, is probably the best estimate for $\mathbf{v}_1$. To compute an estimate for λ_1, examine $A\mathbf{x}_4=\begin{bmatrix}.4594\\.9075\end{bmatrix}$. This vector is approximately $\lambda_1\mathbf{v}_1$. From the second entry in this vector, an estimate of λ_1 is .9075.

4. The vectors in the given sequence approach an eigenvector $\mathbf{v}_1$. The last vector in the sequence, $\mathbf{x}_4=\begin{bmatrix}1\\.7502\end{bmatrix}$, is probably the best estimate for $\mathbf{v}_1$. To compute an estimate for λ_1, examine $A\mathbf{x}_4=\begin{bmatrix}-.4012\\-.3009\end{bmatrix}$. This vector is approximately $\lambda_1\mathbf{v}_1$. From the first entry in this vector, an estimate of λ_1 is $-.4012$.

5. Since $A^5\mathbf{x}=\begin{bmatrix}24991\\-31241\end{bmatrix}$ is an estimate for an eigenvector, the vector $\mathbf{v}=-\frac{1}{31241}\begin{bmatrix}24991\\-31241\end{bmatrix}=\begin{bmatrix}-.7999\\1\end{bmatrix}$ is a vector with a 1 in its second entry that is close to an eigenvector of A. To estimate the dominant eigenvalue λ_1 of A, compute $A\mathbf{v}=\begin{bmatrix}4.0015\\-5.0020\end{bmatrix}$. From the second entry in this vector, an estimate of λ_1 is -5.0020.

6. Since $A^5\mathbf{x}=\begin{bmatrix}-2045\\4093\end{bmatrix}$ is an estimate for an eigenvector, the vector $\mathbf{v}=\frac{1}{4093}\begin{bmatrix}-2045\\4093\end{bmatrix}=\begin{bmatrix}-.4996\\1\end{bmatrix}$ is a vector with a 1 in its second entry that is close to an eigenvector of A. To estimate the dominant eigenvalue λ_1 of A, compute $A\mathbf{v}=\begin{bmatrix}-2.0008\\4.0024\end{bmatrix}$. From the second entry in this vector, an estimate of λ_1 is 4.0024.

7. **[M]** $A=\begin{bmatrix}6&7\\8&5\end{bmatrix}$, $\mathbf{x}_0=\begin{bmatrix}1\\0\end{bmatrix}$. The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}.75\\1\end{bmatrix}$	$\begin{bmatrix}1\\.9565\end{bmatrix}$	$\begin{bmatrix}.9932\\1\end{bmatrix}$	$\begin{bmatrix}1\\.9990\end{bmatrix}$	$\begin{bmatrix}.9998\\1\end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix}6\\8\end{bmatrix}$	$\begin{bmatrix}11.5\\11.0\end{bmatrix}$	$\begin{bmatrix}12.6957\\12.7826\end{bmatrix}$	$\begin{bmatrix}12.9592\\12.9456\end{bmatrix}$	$\begin{bmatrix}12.9927\\12.9948\end{bmatrix}$	$\begin{bmatrix}12.9990\\12.9987\end{bmatrix}$
μ_k	8	11.5	12.7826	12.9592	12.9948	12.9990

The actual eigenvalue is 13.

8. **[M]** $A=\begin{bmatrix}2&1\\4&5\end{bmatrix}$, $\mathbf{x}_0=\begin{bmatrix}1\\0\end{bmatrix}$. The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}.5\\1\end{bmatrix}$	$\begin{bmatrix}.2857\\1\end{bmatrix}$	$\begin{bmatrix}.2558\\1\end{bmatrix}$	$\begin{bmatrix}.2510\\1\end{bmatrix}$	$\begin{bmatrix}.2502\\1\end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix}2\\4\end{bmatrix}$	$\begin{bmatrix}2\\7\end{bmatrix}$	$\begin{bmatrix}1.5714\\6.1429\end{bmatrix}$	$\begin{bmatrix}1.5116\\6.0233\end{bmatrix}$	$\begin{bmatrix}1.5019\\6.0039\end{bmatrix}$	$\begin{bmatrix}1.5003\\6.0006\end{bmatrix}$
μ_k	4	7	6.1429	6.0233	6.0039	6.0006

The actual eigenvalue is 6.

9. [M] $A=\begin{bmatrix}8&0&12\\1&-2&1\\0&3&0\end{bmatrix}, \mathbf{x}_0=\begin{bmatrix}1\\0\\0\end{bmatrix}$. The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4	5	6
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\.125\\0\end{bmatrix}$	$\begin{bmatrix}1\\.0938\\.0469\end{bmatrix}$	$\begin{bmatrix}1\\.1004\\.0328\end{bmatrix}$	$\begin{bmatrix}1\\.0991\\.0359\end{bmatrix}$	$\begin{bmatrix}1\\.0994\\.0353\end{bmatrix}$	$\begin{bmatrix}1\\.0993\\.0354\end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix}8\\1\\0\end{bmatrix}$	$\begin{bmatrix}8\\.75\\.375\end{bmatrix}$	$\begin{bmatrix}8.5625\\.8594\\.2812\end{bmatrix}$	$\begin{bmatrix}8.3942\\.8321\\.3011\end{bmatrix}$	$\begin{bmatrix}8.4304\\.8376\\.2974\end{bmatrix}$	$\begin{bmatrix}8.4233\\.8366\\.2981\end{bmatrix}$	$\begin{bmatrix}8.4246\\.8368\\.2979\end{bmatrix}$
μ_k	8	8	8.5625	8.3942	8.4304	8.4233	8.4246

Thus $\mu_5=8.4233$ and $\mu_6=8.4246$. The actual eigenvalue is $(7+\sqrt{97})/2$, or 8.42443 to five decimal places.

10. [M] $A=\begin{bmatrix}1&2&-2\\1&1&9\\0&1&9\end{bmatrix}, \mathbf{x}_0=\begin{bmatrix}1\\0\\0\end{bmatrix}$. The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4	5	6
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\\0\end{bmatrix}$	$\begin{bmatrix}1\\.6667\\.3333\end{bmatrix}$	$\begin{bmatrix}.3571\\1\\.7857\end{bmatrix}$	$\begin{bmatrix}.0932\\1\\.9576\end{bmatrix}$	$\begin{bmatrix}.0183\\1\\.9904\end{bmatrix}$	$\begin{bmatrix}.0038\\1\\.9982\end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix}1\\1\\0\end{bmatrix}$	$\begin{bmatrix}3\\2\\1\end{bmatrix}$	$\begin{bmatrix}1.6667\\4.6667\\3.6667\end{bmatrix}$	$\begin{bmatrix}.7857\\8.4286\\8.0714\end{bmatrix}$	$\begin{bmatrix}.1780\\9.7119\\9.6186\end{bmatrix}$	$\begin{bmatrix}.0375\\9.9319\\9.9136\end{bmatrix}$	$\begin{bmatrix}.0075\\9.9872\\9.9834\end{bmatrix}$
μ_k	1	3	4.6667	8.4286	9.7119	9.9319	9.9872

Thus $\mu_5=9.9319$ and $\mu_6=9.9872$. The actual eigenvalue is 10.

11. [M] $A=\begin{bmatrix}5&2\\2&2\end{bmatrix}, \mathbf{x}_0=\begin{bmatrix}1\\0\end{bmatrix}$. The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\.4\end{bmatrix}$	$\begin{bmatrix}1\\.4828\end{bmatrix}$	$\begin{bmatrix}1\\.4971\end{bmatrix}$	$\begin{bmatrix}1\\.4995\end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix}5\\2\end{bmatrix}$	$\begin{bmatrix}5.8\\2.8\end{bmatrix}$	$\begin{bmatrix}5.9655\\2.9655\end{bmatrix}$	$\begin{bmatrix}5.9942\\2.9942\end{bmatrix}$	$\begin{bmatrix}5.9990\\2.9990\end{bmatrix}$
μ_k	5	5.8	5.9655	5.9942	5.9990
$R(\mathbf{x}_k)$	5	5.9655	5.9990	5.99997	5.9999993

The actual eigenvalue is 6. The bottom two columns of the table show that $R(\mathbf{x}_k)$ estimates the eigenvalue more accurately than μ_k.

12. [M] $A=\begin{bmatrix}-3&2\\2&2\end{bmatrix}, \mathbf{x}_0=\begin{bmatrix}1\\0\end{bmatrix}$. The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\-.6667\end{bmatrix}$	$\begin{bmatrix}1\\-.4615\end{bmatrix}$	$\begin{bmatrix}1\\-.5098\end{bmatrix}$	$\begin{bmatrix}1\\-.4976\end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix}-3\\2\end{bmatrix}$	$\begin{bmatrix}-4.3333\\2.0000\end{bmatrix}$	$\begin{bmatrix}-3.9231\\2.0000\end{bmatrix}$	$\begin{bmatrix}-4.0196\\2.0000\end{bmatrix}$	$\begin{bmatrix}-3.9951\\2.0000\end{bmatrix}$
μ_k	−3	−4.3333	−3.9231	−4.0196	−3.9951
$R(\mathbf{x}_k)$	−3	−3.9231	−3.9951	−3.9997	−3.99998

The actual eigenvalue is −4. The bottom two columns of the table show that $R(\mathbf{x}_k)$ estimates the eigenvalue more accurately than μ_k.

13. If the eigenvalues close to 4 and −4 have different absolute values, then one of these is a strictly dominant eigenvalue, so the power method will work. But the power method depends on powers of the quotients λ_2/λ_1 and λ_3/λ_1 going to zero. If $|\lambda_2/\lambda_1|$ is close to 1, its powers will go to zero slowly, and the power method will converge slowly.

14. If the eigenvalues close to 4 and −4 have the same absolute value, then neither of these is a strictly dominant eigenvalue, so the power method will not work. However, the inverse power method may still be used. If the initial estimate is chosen near the eigenvalue close to 4, then the inverse power method should produce a sequence that estimates the eigenvalue close to 4.

15. Suppose $A\mathbf{x}=\lambda\mathbf{x}$, with $\mathbf{x}\neq 0$. For any α, $A\mathbf{x}-\alpha I\mathbf{x}=(\lambda-\alpha)\mathbf{x}$. If α is *not* an eigenvalue of A, then $A-\alpha I$ is invertible and $\lambda-\alpha$ is not 0; hence

$$\mathbf{x}=(A-\alpha I)^{-1}(\lambda-\alpha)\mathbf{x} \text{ and } (\lambda-\alpha)^{-1}\mathbf{x}=(A-\alpha I)^{-1}\mathbf{x}$$

This last equation shows that $\mathbf{x}$ is an eigenvector of $(A-\alpha I)^{-1}$ corresponding to the eigenvalue $(\lambda-\alpha)^{-1}$.

16. Suppose that μ is an eigenvalue of $(A-\alpha I)^{-1}$ with corresponding eigenvector $\mathbf{x}$. Since $(A-\alpha I)^{-1}\mathbf{x}=\mu\mathbf{x}$,

$$\mathbf{x}=(A-\alpha I)(\mu\mathbf{x})=A(\mu\mathbf{x})-(\alpha I)(\mu\mathbf{x})=\mu(A\mathbf{x})-\alpha\mu\mathbf{x}$$

Solving this equation for $A\mathbf{x}$, we find that

$$A\mathbf{x} = \left(\frac{1}{\mu}\right)(\alpha\mu\mathbf{x} + \mathbf{x}) = \left(\alpha + \frac{1}{\mu}\right)\mathbf{x}$$

Thus $\lambda = \alpha + (1/\mu)$ is an eigenvalue of A with corresponding eigenvector $\mathbf{x}$.

17. [M] $A = \begin{bmatrix} 10 & -8 & -4 \\ -8 & 13 & 4 \\ -4 & 5 & 4 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \alpha = 3.3.$ The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .7873 \\ .0908 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .7870 \\ .0957 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 26.0552 \\ 20.5128 \\ 2.3669 \end{bmatrix}$	$\begin{bmatrix} 47.1975 \\ 37.1436 \\ 4.5187 \end{bmatrix}$	$\begin{bmatrix} 47.1233 \\ 37.0866 \\ 4.5083 \end{bmatrix}$
μ_k	26.0552	47.1975	47.1233
ν_k	3.3384	3.32119	3.3212209

Thus an estimate for the eigenvalue to four decimal places is 3.3212. The actual eigenvalue is $(25-\sqrt{337})/2$, or 3.3212201 to seven decimal places.

18. [M] $A = \begin{bmatrix} 8 & 0 & 12 \\ 1 & -2 & 1 \\ 0 & 3 & 0 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \alpha = -1.4.$ The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .3646 \\ -.7813 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .3734 \\ -.7855 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .3729 \\ -.7854 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .3729 \\ -.7854 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 40 \\ 14.5833 \\ -31.25 \end{bmatrix}$	$\begin{bmatrix} -38.125 \\ -14.2361 \\ 29.9479 \end{bmatrix}$	$\begin{bmatrix} -41.1134 \\ -15.3300 \\ 32.2888 \end{bmatrix}$	$\begin{bmatrix} -40.9243 \\ -15.2608 \\ 32.1407 \end{bmatrix}$	$\begin{bmatrix} -40.9358 \\ -15.2650 \\ 32.1497 \end{bmatrix}$
μ_k	40	−38.125	−41.1134	−40.9243	−40.9358
ν_k	−1.375	−1.42623	−1.42432	−1.42444	−1.42443

Thus an estimate for the eigenvalue to four decimal places is −1.4244. The actual eigenvalue is $(7-\sqrt{97})/2$, or −1.424429 to six decimal places.

19. [M] $A = \begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

(a) The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .7 \\ .8 \\ .7 \end{bmatrix}$	$\begin{bmatrix} .988679 \\ .709434 \\ 1 \\ .932075 \end{bmatrix}$	$\begin{bmatrix} .961467 \\ .691491 \\ 1 \\ .942201 \end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix} 10 \\ 7 \\ 8 \\ 7 \end{bmatrix}$	$\begin{bmatrix} 26.2 \\ 18.8 \\ 26.5 \\ 24.7 \end{bmatrix}$	$\begin{bmatrix} 29.3774 \\ 21.1283 \\ 30.5547 \\ 28.7887 \end{bmatrix}$	$\begin{bmatrix} 29.0505 \\ 20.8987 \\ 30.3205 \\ 28.6097 \end{bmatrix}$
μ_k	10	26.5	30.5547	30.3205

k	4	5	6	7
$\mathbf{x}_k$	$\begin{bmatrix} .958115 \\ .689261 \\ 1 \\ .943578 \end{bmatrix}$	$\begin{bmatrix} .957691 \\ .688978 \\ 1 \\ .943755 \end{bmatrix}$	$\begin{bmatrix} .957637 \\ .688942 \\ 1 \\ .943778 \end{bmatrix}$	$\begin{bmatrix} .957630 \\ .688938 \\ 1 \\ .943781 \end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix} 29.0110 \\ 20.8710 \\ 30.2927 \\ 28.5889 \end{bmatrix}$	$\begin{bmatrix} 29.0060 \\ 20.8675 \\ 30.2892 \\ 28.5863 \end{bmatrix}$	$\begin{bmatrix} 29.0054 \\ 20.8671 \\ 30.2887 \\ 28.5859 \end{bmatrix}$	$\begin{bmatrix} 29.0053 \\ 20.8670 \\ 30.2887 \\ 28.5859 \end{bmatrix}$
μ_k	30.2927	30.2892	30.2887	30.2887

Thus an estimate for the eigenvalue to four decimal places is 30.2887. The actual eigenvalue is 30.2886853 to seven decimal places. An estimate for the corresponding eigenvector is

$$\begin{bmatrix} .957630 \\ .688938 \\ 1 \\ .943781 \end{bmatrix}.$$

(b) The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}-.609756\\1\\-.243902\\.146341\end{bmatrix}$	$\begin{bmatrix}-.604007\\1\\-.251051\\.148899\end{bmatrix}$	$\begin{bmatrix}-.603973\\1\\-.251134\\.148953\end{bmatrix}$	$\begin{bmatrix}-.603972\\1\\-.251135\\.148953\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}25\\-41\\10\\-6\end{bmatrix}$	$\begin{bmatrix}-59.5610\\98.6098\\-24.7561\\14.6829\end{bmatrix}$	$\begin{bmatrix}-59.5041\\98.5211\\-24.7420\\14.6750\end{bmatrix}$	$\begin{bmatrix}-59.5044\\98.5217\\-24.7423\\14.6751\end{bmatrix}$	$\begin{bmatrix}-59.5044\\98.5217\\-24.7423\\14.6751\end{bmatrix}$
μ_k	−41	98.6098	98.5211	98.5217	98.5217
ν_k	−.0243902	.0101410	.0101501	.0101500	.0101500

Thus an estimate for the eigenvalue to five decimal places is .01015. The actual eigenvalue is .01015005 to eight decimal places. An estimate for the corresponding eigenvector is

$$\begin{bmatrix}-.603972\\1\\-.251135\\.148953\end{bmatrix}.$$

20. [M] $A=\begin{bmatrix}1&2&3&2\\2&12&13&11\\-2&3&0&2\\4&5&7&2\end{bmatrix}, \mathbf{x}_0=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}.$

(a) The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}.25\\.5\\-.5\\1\end{bmatrix}$	$\begin{bmatrix}.159091\\1\\.272727\\.181818\end{bmatrix}$	$\begin{bmatrix}.187023\\1\\.170483\\.442748\end{bmatrix}$	$\begin{bmatrix}.184166\\1\\.180439\\.402197\end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix}1\\2\\-2\\4\end{bmatrix}$	$\begin{bmatrix}1.75\\11\\3\\2\end{bmatrix}$	$\begin{bmatrix}3.34091\\17.8636\\3.04545\\7.90909\end{bmatrix}$	$\begin{bmatrix}3.58397\\19.4606\\3.51145\\7.82697\end{bmatrix}$	$\begin{bmatrix}3.52988\\19.1382\\3.43606\\7.80413\end{bmatrix}$
μ_k	4	11	17.8636	19.4606	19.1382

k	5	6	7	8	9
$\mathbf{x}_k$	$\begin{bmatrix}.184441\\1\\.179539\\.407778\end{bmatrix}$	$\begin{bmatrix}.184414\\1\\.179622\\.407021\end{bmatrix}$	$\begin{bmatrix}.184417\\1\\.179615\\.407121\end{bmatrix}$	$\begin{bmatrix}.184416\\1\\.179615\\.407108\end{bmatrix}$	$\begin{bmatrix}.184416\\1\\.179615\\.407110\end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix}3.53861\\19.1884\\3.44667\\7.81010\end{bmatrix}$	$\begin{bmatrix}3.53732\\19.1811\\3.44521\\7.80905\end{bmatrix}$	$\begin{bmatrix}3.53750\\19.1822\\3.44541\\7.80921\end{bmatrix}$	$\begin{bmatrix}3.53748\\19.1820\\3.44538\\7.80919\end{bmatrix}$	$\begin{bmatrix}3.53748\\19.1820\\3.44539\\7.80919\end{bmatrix}$
μ_k	19.1884	19.1811	19.1822	19.1820	19.1820

Thus an estimate for the eigenvalue to four decimal places is 19.1820. The actual eigenvalue is 19.1820368 to seven decimal places. An estimate for the corresponding eigenvector is

$$\begin{bmatrix}.184416\\1\\.179615\\.407110\end{bmatrix}.$$

(b) The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\.226087\\-.921739\\.660870\end{bmatrix}$	$\begin{bmatrix}1\\.222577\\-.917970\\.660496\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}115\\26\\-106\\76\end{bmatrix}$	$\begin{bmatrix}81.7304\\18.1913\\-75.0261\\53.9826\end{bmatrix}$	$\begin{bmatrix}81.9314\\18.2387\\-75.2125\\54.1143\end{bmatrix}$
μ_k	115	81.7304	81.9314
ν_k	.00869565	.0122353	.0122053

Thus an estimate for the eigenvalue to four decimal places is .0122. The actual eigenvalue is .01220556 to eight decimal places. An estimate for the corresponding eigenvector is

$$\begin{bmatrix}1\\.222577\\-.917970\\.660496\end{bmatrix}.$$

21. (a) $A=\begin{bmatrix}.8 & 0\\ 0 & .2\end{bmatrix}, \mathbf{x}=\begin{bmatrix}.5\\ .5\end{bmatrix}$. Here is the sequence $A^k\mathbf{x}$ for $k=1,\ldots 5$:

$$\begin{bmatrix}.4\\ .1\end{bmatrix}, \begin{bmatrix}.32\\ .02\end{bmatrix}, \begin{bmatrix}.256\\ .004\end{bmatrix}, \begin{bmatrix}.2048\\ .0008\end{bmatrix}, \begin{bmatrix}.16384\\ .00016\end{bmatrix}$$

Notice that $A^5\mathbf{x}$ is approximately $.8(A^4\mathbf{x})$.

Conclusion: If the eigenvalues of A are all less than 1 in magnitude, and if $\mathbf{x}\neq 0$, then $A^k\mathbf{x}$ is approximately an eigenvector for large k.

(b) $A=\begin{bmatrix}1 & 0\\ 0 & .8\end{bmatrix}, \mathbf{x}=\begin{bmatrix}.5\\ .5\end{bmatrix}$. Here is the sequence $A^k\mathbf{x}$ for $k=1,\ldots 5$:

$$\begin{bmatrix}.5\\ .4\end{bmatrix}, \begin{bmatrix}.5\\ .32\end{bmatrix}, \begin{bmatrix}.5\\ .256\end{bmatrix}, \begin{bmatrix}.5\\ .2048\end{bmatrix}, \begin{bmatrix}.5\\ .16384\end{bmatrix}$$

Notice that $A^k\mathbf{x}$ seems to be converging to $\begin{bmatrix}.5\\ 0\end{bmatrix}$.

Conclusion: If the strictly dominant eigenvalue of A is 1, and if $\mathbf{x}$ has a component in the direction of the corresponding eigenvector, then $\{A^k\mathbf{x}\}$ will converge to a multiple of that eigenvector.

c. $A=\begin{bmatrix}8 & 0\\ 0 & 2\end{bmatrix}, \mathbf{x}=\begin{bmatrix}.5\\ .5\end{bmatrix}$. Here is the sequence $A^k\mathbf{x}$ for $k=1,\ldots 5$:

$$\begin{bmatrix}4\\ 1\end{bmatrix}, \begin{bmatrix}32\\ 2\end{bmatrix}, \begin{bmatrix}256\\ 4\end{bmatrix}, \begin{bmatrix}2048\\ 8\end{bmatrix}, \begin{bmatrix}16384\\ 16\end{bmatrix}$$

Notice that the distance of $A^k\mathbf{x}$ from either eigenvector of A is increasing rapidly as k increases.

Conclusion: If the eigenvalues of A are all greater than 1 in magnitude, and if $\mathbf{x}$ is not an eigenvector, then the distance from $A^k\mathbf{x}$ to the nearest eigenvector will *increase* as $k\to\infty$.

Chapter 5 SUPPLEMENTARY EXERCISES

1. **a.** True. If A is invertible and if $A\mathbf{x}=1\cdot\mathbf{x}$ for some nonzero $\mathbf{x}$, then left-multiply by A^{-1} to obtain $\mathbf{x}=A^{-1}\mathbf{x}$, which may be rewritten as $A^{-1}\mathbf{x}=1\cdot\mathbf{x}$. Since $\mathbf{x}$ is nonzero, this shows 1 is an eigenvalue of A^{-1}.

b. False. If A is row equivalent to the identity matrix, then A is invertible. The matrix in Example 4 of Section 5.3 shows that an invertible matrix need not be diagonalizable. Also, see Exercise 31 in Section 5.3.

c. True. If A contains a row or column of zeros, then A is not row equivalent to the identity matrix and thus is not invertible. By the Invertible Matrix Theorem (as stated in Section 5.2), 0 is an eigenvalue of A.

d. False. Consider a diagonal matrix D whose eigenvalues are 1 and 3, that is, its diagonal entries are 1 and 3. Then D^2 is a diagonal matrix whose eigenvalues (diagonal entries) are 1 and 9. In general, the eigenvalues of A^2 are the *squares* of the eigenvalues of A.

e. True. Suppose a nonzero vector $\mathbf{x}$ satisfies $A\mathbf{x}=\lambda\mathbf{x}$, then

$$A^2\mathbf{x}=A(A\mathbf{x})=A(\lambda\mathbf{x})=\lambda A\mathbf{x}=\lambda^2\mathbf{x}$$

This shows that $\mathbf{x}$ is also an eigenvector for A^2

f. True. Suppose a nonzero vector $\mathbf{x}$ satisfies $A\mathbf{x}=\lambda\mathbf{x}$, then left-multiply by A^{-1} to obtain $\mathbf{x}=A^{-1}(\lambda\mathbf{x})=\lambda A^{-1}\mathbf{x}$. Since A is invertible, the eigenvalue λ is not zero. So $\lambda^{-1}\mathbf{x}=A^{-1}\mathbf{x}$, which shows that $\mathbf{x}$ is also an eigenvector of A^{-1}.

g. False. Zero is an eigenvalue of each singular square matrix.

h. True. By definition, an eigenvector must be nonzero.

i. False. Let $A=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$ then $\mathbf{e}_1=\begin{bmatrix}1\\ 0\end{bmatrix}$ and $\mathbf{e}_2=\begin{bmatrix}0\\ 1\end{bmatrix}$ are eigenvectors of A for the eigenvalue 2, and they are linearly independent.

j. True. This follows from Theorem 4 in Section 5.2

k. False. Let A be the 3×3 matrix in Example 3 of Section 5.3. Then A is similar to a diagonal matrix D. The eigenvectors of D are the columns of I_3, but the eigenvectors of A are entirely different.

l. False. Let $A=\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$. Then $\mathbf{e}_1=\begin{bmatrix}1\\ 0\end{bmatrix}$ and $\mathbf{e}_2=\begin{bmatrix}0\\ 1\end{bmatrix}$ are eigenvectors of A, but $\mathbf{e}_1+\mathbf{e}_2$ is not.

(Actually, it can be shown that if two eigenvectors of A correspond to distinct eigenvalues, then their sum cannot be an eigenvector.)

m. False. *All* the diagonal entries of an upper triangular matrix are the eigenvalues of the matrix (Theorem 1 in Section 5.1). A diagonal entry may be zero.

n. True. Matrices A and A^T have the same characteristic polynomial, because $\det(A^T-\lambda I)=\det(A-\lambda I)^T=\det(A-\lambda I)$, by the determinant transpose property.

o. False. Counterexample: Let A be the 5×5 identity matrix.

p. True. For example, let A be the matrix that rotates vectors through $\pi/2$ radians about the origin. Then $A\mathbf{x}$ is not a multiple of $\mathbf{x}$ when $\mathbf{x}$ is nonzero.

q. False. If A is a diagonal matrix with 0 on the diagonal, then the columns of A are not linearly independent.

r. True. If $A\mathbf{x}=\lambda_1\mathbf{x}$ and $A\mathbf{x}=\lambda_2\mathbf{x}$, then $\lambda_1\mathbf{x}=\lambda_2\mathbf{x}$ and $(\lambda_1-\lambda_2)\mathbf{x}=\mathbf{0}$. If $\mathbf{x}\neq\mathbf{0}$, then λ_1 must equal λ_2.

s. False. Let A be a singular matrix that is diagonalizable. (For instance, let A be a diagonal matrix with 0 on the diagonal.) Then, by Theorem 8 in Section 5.4, the transformation $\mathbf{x}\mapsto A\mathbf{x}$ is represented by a diagonal matrix relative to a coordinate system determined by eigenvectors of A.

t. True. By definition of matrix multiplication,

$A = AI = A[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n] = [A\mathbf{e}_1 \quad A\mathbf{e}_2 \quad \cdots \quad A\mathbf{e}_n]$

If $A\mathbf{e}_j = d_j\mathbf{e}_j$ for $j = 1,\ldots,n$, then A is a diagonal matrix with diagonal entries $d_1,\ldots,d_n$.

u. True. If $B = PDP^{-1}$, where D is a diagonal matrix, and if $A = QBQ^{-1}$, then $A = Q(PDP^{-1})Q^{-1} = (QP)D(QP)^{-1}$, which shows that A is diagonalizable.

v. True. Since B is invertible, AB is similar to $B(AB)B^{-1}$, which equals BA.

w. False. Having n linearly independent eigenvectors makes an $n\times n$ matrix diagonalizable (by the Diagonalization Theorem 5 in Section 5.3), but not necessarily invertible. One of the eigenvalues of the matrix could be zero.

x. True. If A is diagonalizable, then by the Diagonalization Theorem, A has n linearly independent eigenvectors $\mathbf{v}_1,\ldots,\mathbf{v}_n$ in $\mathbf{R}^n$. By the Basis Theorem, $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ spans $\mathbf{R}^n$. This means that each vector in $\mathbf{R}^n$ can be written as a linear combination of $\mathbf{v}_1,\ldots,\mathbf{v}_n$.

2. Suppose $B\mathbf{x} \neq \mathbf{0}$ and $AB\mathbf{x} = \lambda\mathbf{x}$ for some λ. Then $A(B\mathbf{x}) = \lambda\mathbf{x}$. Left-multiply each side by B and obtain $BA(B\mathbf{x}) = B(\lambda\mathbf{x}) = \lambda(B\mathbf{x})$. This equation says that $B\mathbf{x}$ is an eigenvector of BA, because $B\mathbf{x} \neq \mathbf{0}$.

3. a. Suppose $A\mathbf{x} = \lambda\mathbf{x}$, with $\mathbf{x} \neq \mathbf{0}$. Then $(5I - A)\mathbf{x} = 5\mathbf{x} - A\mathbf{x} = 5\mathbf{x} - \lambda\mathbf{x} = (5-\lambda)\mathbf{x}$. The eigenvalue is $5-\lambda$.

b. $(5I - 3A + A^2)\mathbf{x} = 5\mathbf{x} - 3A\mathbf{x} + A(A\mathbf{x}) = 5\mathbf{x} - 3(\lambda\mathbf{x}) + \lambda^2\mathbf{x} = (5 - 3\lambda + \lambda^2)\mathbf{x}$. The eigenvalue is $5 - 3\lambda + \lambda^2$.

4. Assume that $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x}$. The desired statement is true for $m = 1$, by the assumption about λ. Suppose that for some $k \geq 1$, the statement holds when $m = k$. That is, suppose that $A^k\mathbf{x} = \lambda^k\mathbf{x}$. Then $A^{k+1}\mathbf{x} = A(A^k\mathbf{x}) = A(\lambda^k\mathbf{x})$ by the induction hypothesis. Continuing, $A^{k+1}\mathbf{x} = \lambda^k A\mathbf{x} = \lambda^{k+1}\mathbf{x}$, because $\mathbf{x}$ is an eigenvector of A corresponding to A. Since $\mathbf{x}$ is nonzero, this equation shows that λ^{k+1} is an eigenvalue of A^{k+1}, with corresponding eigenvector $\mathbf{x}$. Thus the desired statement is true when $m = k+1$. By the principle of induction, the statement is true for each positive integer m.

5. Suppose $A\mathbf{x} = \lambda\mathbf{x}$, with $\mathbf{x} \neq \mathbf{0}$. Then

$$\begin{aligned} p(A)\mathbf{x} &= (c_0I + c_1A + c_2A^2 + \ldots + c_nA^n)\mathbf{x} \\ &= c_0\mathbf{x} + c_1A\mathbf{x} + c_2A^2\mathbf{x} + \ldots + c_nA^n\mathbf{x} \\ &= c_0\mathbf{x} + c_1\lambda\mathbf{x} + c_2\lambda^2\mathbf{x} + \ldots + c_n\lambda^n\mathbf{x} = p(\lambda)\mathbf{x} \end{aligned}$$

So $p(\lambda)$ is an eigenvalue of $p(A)$.

6. a. If $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$, and

$$\begin{aligned} B &= 5I - 3A + A^2 = 5PIP^{-1} - 3PDP^{-1} + PD^2P^{-1} \\ &= P(5I - 3D + D^2)P^{-1} \end{aligned}$$

Since D is diagonal, so is $5I - 3D + D^2$. Thus B is similar to a diagonal matrix.

b.
$$\begin{aligned} p(A) &= c_0I + c_1PDP^{-1} + c_2PD^2P^{-1} + \cdots + c_nPD^nP^{-1} \\ &= P(c_0I + c_1D + c_2D^2 + \cdots + c_nD^n)P^{-1} \\ &= Pp(D)P^{-1} \end{aligned}$$

This shows that $p(A)$ is diagonalizable, because $p(D)$ is a linear combination of diagonal matrices and hence is diagonal. In fact, because D is diagonal, it is easy to see that

$$p(D) = \begin{bmatrix} p(2) & 0 \\ 0 & p(7) \end{bmatrix}$$

7. If $A = PDP^{-1}$, then $p(A) = Pp(D)P^{-1}$, as shown in Exercise 6. If the (j, j) entry in D is λ, then the (j, j) entry in D^k is λ^k, and so the (j, j) entry in $p(D)$ is $p(\lambda)$. If p is the characteristic polynomial of A, then $p(\lambda) = 0$ for each diagonal entry of D, because these entries in D are the eigenvalues of A. Thus $p(D)$ is the zero matrix. Thus $p(A) = P\cdot 0\cdot P^{-1} = 0$.

8. a. If λ is an eigenvalue of an $n\times n$ diagonalizable matrix A, then $A = PDP^{-1}$ for an invertible matrix P and an $n\times n$ diagonal matrix D whose diagonal entries are the eigenvalues of A. If the multiplicity of λ is n, then λ must appear in every diagonal entry of D. That is, $D = \lambda I$. In this case, $A = P(\lambda I)P^{-1} = \lambda PIP^{-1} = \lambda PP^{-1} = \lambda I$.

b. Since the matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is triangular, its eigenvalues are on the diagonal. Thus 3 is an eigenvalue with multiplicity 2. If the 2×2 matrix A were diagonalizable, then A would be $3I$, by part (a). This is not the case, so A is not diagonalizable.

9. If $I - A$ were not invertible, then the equation $(I - A)\mathbf{x} = \mathbf{0}$. would have a nontrivial solution $\mathbf{x}$. Then $\mathbf{x} - A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = 1\cdot\mathbf{x}$, which shows that A would have 1 as an eigenvalue. This cannot happen if all the eigenvalues are less than 1 in magnitude. So $I - A$ must be invertible.

10. To show that A^k tends to the zero matrix, it suffices to show that each column of A^k can be made as close to the zero vector as desired by taking k sufficiently large. The jth column of A is $A\mathbf{e}_j$, where $\mathbf{e}_j$ is the jth column of the identity matrix. Since A is diagonalizable, there is a basis for $\mathbb{R}^n$ consisting of eigenvectors $\mathbf{v}_1,\ldots,\mathbf{v}_n$, corresponding to eigenvalues $\lambda_1,\ldots,\lambda_n$. So there exist scalars $c_1,\ldots,c_n$, such that

$\mathbf{e}_j = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ (an eigenvector decomposition of $\mathbf{e}_j$)

Then, for $k = 1, 2,\ldots$,

$$A^k\mathbf{e}_j = c_1(\lambda_1)^k\mathbf{v}_1 + \cdots + c_n(\lambda_n)^k\mathbf{v}_n \qquad (*)$$

If the eigenvalues are all less than 1 in absolute value, then their kth powers all tend to zero. So (*) shows that $A^k\mathbf{e}_j$ tends to the zero vector, as desired.

11. a. Take $\mathbf{x}$ in H. Then $\mathbf{x} = c\mathbf{u}$ for some scalar c. So $A\mathbf{x} = A(c\mathbf{u}) = c(A\mathbf{u}) = c(\lambda\mathbf{u}) = (c\lambda)\mathbf{u}$, which shows that $A\mathbf{x}$ is in H.

b. Let **x** be a nonzero vector in K. Since K is one-dimensional, K must be the set of all scalar multiples of **x**. If K is invariant under A, then $A\mathbf{x}$ is in K and hence $A\mathbf{x}$ is a multiple of **x**. Thus **x** is an eigenvector of A.

12. Let U and V be echelon forms of A and B, obtained with r and s row interchanges, respectively, and no scaling. Then $\det A = (-1)^r \det U$ and $\det B = (-1)^s \det V$

Using first the row operations that reduce A to U, we can reduce G to a matrix of the form $G' = \begin{bmatrix} U & Y \\ 0 & B \end{bmatrix}$. Then, using the row operations that reduce B to V, we can further reduce G' to $G'' = \begin{bmatrix} U & Y \\ 0 & V \end{bmatrix}$. There will be $r+s$ row interchanges, and so

$\det G = \det \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} = (-1)^{r+s} \det \begin{bmatrix} U & Y \\ 0 & V \end{bmatrix}$ Since $\begin{bmatrix} U & Y \\ 0 & V \end{bmatrix}$ is upper triangular, its determinant equals the product of the diagonal entries,
and since U and V are upper triangular, this product also equals (det U) (det V). Thus

$\det G = (-1)^{r+s}(\det U)(\det V) = (\det A)(\det B)$

For any scalar λ, the matrix $G - \lambda I$ has the same partitioned form as G, with $A - \lambda I$ and $B - \lambda I$ as its diagonal blocks. (Here I represents various identity matrices of appropriate sizes.) Hence the result about det G shows that $\det(G - \lambda I) = \det(A - \lambda I) \cdot \det(B - \lambda I)$

13. By Exercise 12, the eigenvalues of A are the eigenvalues of the matrix $[3]$ together with the eigenvalues of $\begin{bmatrix} 5 & -2 \\ -4 & 3 \end{bmatrix}$. The only eigenvalue of $[3]$ is 3, while the eigenvalues of $\begin{bmatrix} 5 & -2 \\ -4 & 3 \end{bmatrix}$ are 1 and 7. Thus the eigenvalues of A are 1, 3, and 7.

14. By Exercise 12, the eigenvalues of A are the eigenvalues of the matrix $\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$ together with the eigenvalues of $\begin{bmatrix} -7 & -4 \\ 3 & 1 \end{bmatrix}$. The eigenvalues of $\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$ are -1 and 6, while the eigenvalues of $\begin{bmatrix} -7 & -4 \\ 3 & 1 \end{bmatrix}$ are -5 and -1. Thus the eigenvalues of A are $-1, -5$, and 6, and the eigenvalue -1 has multiplicity 2.

15. Replace a by $a - \lambda$ in the determinant formula from Exercise 16 in Chapter 3 Supplementary Exercises.

$\det(A - \lambda I) = (a - b - \lambda)^{n-1}[a - \lambda + (n-1)b]$

This determinant is zero only if $a - b - \lambda = 0$ or $a - \lambda + (n-1)b = 0$. Thus λ is an eigenvalue of A if and only if $\lambda = a - b$ or $\lambda = a + (n-1)$. From the formula for $\det(A - \lambda I)$ above, the algebraic multiplicity is $n-1$ for $a-b$ and 1 for $a+(n-1)b$.

16. The 3×3 matrix has eigenvalues $1-2$ and $1+(2)(2)$, that is, -1 and 5. The eigenvalues of the 5×5 matrix are $7-3$ and $7+(4)(3)$, that is 4 and 19.

17. Note that $\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$
$= \lambda^2 - (\operatorname{tr} A)\lambda + \det A$, and use the quadratic formula to solve the characteristic equation:

$$\lambda = \frac{\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4\det A}}{2}$$

The eigenvalues are both real if and only if the discriminant is nonnegative, that is, $(\operatorname{tr} A)^2 - 4\det A \ge 0$. This inequality simplifies to $(\operatorname{tr} A)^2 \ge 4\det A$ and $\left(\frac{trA}{2}\right)^2 \ge \det A$.

18. The eigenvalues of A are 1 and .6. Use this to factor A and A^k.

$$A = \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & .6 \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 2 & 3 \\ -2 & -1 \end{bmatrix}$$

$$A^k = \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1^k & 0 \\ 0 & .6^k \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 2 & 3 \\ -2 & -1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -2\cdot(.6)^k & -(.6)^k \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -2+6(.6)^k & -3+3(.6)^k \\ 4-4(.6)^k & 6-2(.6)^k \end{bmatrix}$$

$$\to \frac{1}{4}\begin{bmatrix} -2 & -3 \\ 4 & 6 \end{bmatrix} \text{ as } k \to \infty$$

19. $C_p = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}; \det(C_p - \lambda I) = 6 - 5\lambda + \lambda^2 = p(\lambda)$

20. $C_p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 24 & -26 & 9 \end{bmatrix};$

$\det(C_p - \lambda I) = 24 - 26\lambda + 9\lambda^2 - \lambda^3 = p(\lambda)$

21. If p is a polynomial of order 2, then a calculation such as in Exercise 19 shows that the characteristic polynomial of C_p is $p(\lambda) = (-1)^2 p(\lambda)$, so the result is true for $n=2$. Suppose the result is true for $n=k$ for some $k \ge 2$, and consider a polynomial p of degree $k+1$. Then expanding $\det(C_p - \lambda I)$ by cofactors down the first column, the determinant of $C_p - \lambda I$ equals

$$(-\lambda)\det\begin{bmatrix} -\lambda & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & & 1 \\ -a_1 & -a_2 & \cdots & -a_k - \lambda \end{bmatrix} + (-1)^{k+1}a_0$$

The $k \times k$ matrix shown is $C_q - \lambda I$, where $q(t) = a_1 + a_2 t + \cdots + a_k t^{k-1} + t^k$. By the induction assumption, the determinant of $C_q - \lambda I$ is $(-1)^k q(\lambda)$. Thus

$$\begin{aligned}\det(C_p - \lambda I) &= (-1)^{k+1} a_0 + (-\lambda)(-1)^k q(\lambda) \\ &= (-1)^{k+1}[a_0 + \lambda(a_1 + \cdots + a_k \lambda^{k-1} + \lambda^k)] \\ &= (-1)^{k+1} p(\lambda)\end{aligned}$$

So the formula holds for $n = k+1$ when it holds for $n = k$. By the principle of induction, the formula for $\det(C_p - \lambda I)$ is true for all $n \geq 2$.

22. **a.** $C_p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}$

b. Since λ is a zero of p, $a_0 + a_1\lambda + a_2\lambda^2 + \lambda^3 = 0$ and $-a_0 - a_1\lambda - a_2\lambda^2 = \lambda^3$. Thus

$$C_p \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ -a_0 - a_1\lambda - a_2\lambda^2 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ \lambda^3 \end{bmatrix}$$

That is, $C_p(1, \lambda, \lambda^2) = \lambda(1, \lambda, \lambda^2)$, which shows that $(1, \lambda, \lambda^2)$ is an eigenvector of C_p corresponding to the eigenvalue λ.

23. From Exercise 22, the columns of the Vandermonde matrix *V* are eigenvectors of C_p, corresponding to the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ (the roots of the polynomial *p*). Since these eigenvalues are distinct, the eigenvectors from a linearly independent set, by Theorem 2 in Section 5.1. Thus *V* has linearly independent columns and hence is invertible, by the Invertible Matrix Theorem. Finally, since the columns of *V* are eigenvectors of C_p, the Diagonalization Theorem (Theorem 5 in Section 5.3) shows that $V^{-1}C_pV$ is diagonal.

24. **[M]** The MATLAB command `roots(p)` requires as input a row vector *p* whose entries are the coefficients of a polynomial, with the highest order coefficient listed first. MATLAB constructs a companion matrix C_p whose characteristic polynomial is *p*, so the roots of *p* are the eigenvalues of C_p. The numerical values of the eigenvalues (roots) are found by the same QR algorithm used by the command `eig(A)`.

25. **[M]** The MATLAB command `[P D]=eig(A)` produces a matrix *P*, whose condition number is 1.6×10^8, and a diagonal matrix *D*, whose entries are *almost* 2, 2, 1. However, the exact eigenvalues of *A* are 2, 2, 1, and *A* is not diagonalizable.

26. **[M]** This matrix may cause the same sort of trouble as the matrix in Exercise 25. A matrix program that computes eigenvalues by an interative process may indicate that *A* has four distinct eigenvalues, all close to zero. However, the only eigenvalue is 0, with multiplicity 4, because $A^4 = 0$.