$\mathbf{x}_0 = \mathbf{v}_1 + .1\mathbf{v}_2 + .3\mathbf{v}_3$

$\mathbf{x}_1 = A\mathbf{v}_1 + .1A\mathbf{v}_2 + .3A\mathbf{v}_3 = \mathbf{v}_1 + .1(.5)\mathbf{v}_2 + .3(.2)\mathbf{v}_3$, and

$\mathbf{x}_k = \mathbf{v}_1 + .1(.5)^k\mathbf{v}_2 + .3(.2)^k\mathbf{v}_3$. As $k$ increases, $\mathbf{x}_k$ approaches $\mathbf{v}_1$.

**16. [M]**

$$A = \begin{bmatrix} .90 & .01 & .09 \\ .01 & .90 & .01 \\ .09 & .09 & .90 \end{bmatrix} \cdot \texttt{ev = eig(A)} = \begin{bmatrix} 1.0000 \\ 0.8900 \\ .8100 \end{bmatrix}. \text{ To four decimal places,}$$

$$\mathbf{v}_1 = \texttt{nulbasis(A-eye(3))} = \begin{bmatrix} 0.9192 \\ 0.1919 \\ 1.0000 \end{bmatrix}. \text{ Exact: } \begin{bmatrix} 91/99 \\ 19/99 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \texttt{nulbasis(A-ev(2)*eye(3))} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_3 = \texttt{nulbasis(A-ev(3)*eye(3))} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The general solution of the dynamical system is $\mathbf{x}_k = c_1\mathbf{v}_1 + c_2(.89)^k\mathbf{v}_2 + c_3(.81)^k\mathbf{v}_3$.

**Note:** When working with stochastic matrices and starting with a probability vector (having nonnegative entries whose sum is 1), it helps to scale $\mathbf{v}_1$ to make its entries sum to 1. If $\mathbf{v}_1 = (91/209, 19/209, 99/209)$, or $(.435, .091, .474)$ to three decimal places, then the weight $c_1$ above turns out to be 1. See the text's discussion of Exercise 27 in Section 5.2.

**17. a.** $A = \begin{bmatrix} 0 & 1.6 \\ .3 & .8 \end{bmatrix}$

**b.** $\det\begin{bmatrix} -\lambda & 1.6 \\ .3 & .8-\lambda \end{bmatrix} = \lambda^2 - .8\lambda - .48 = 0$. The eigenvalues of $A$ are given by

$$\lambda = \frac{.8 \pm \sqrt{(-.8)^2 - 4(-.48)}}{2} = \frac{.8 \pm \sqrt{2.56}}{2} = \frac{.8 \pm 1.6}{2} = 1.2 \text{ and } -.4$$

The numbers of juveniles and adults are increasing because the largest eigenvalue is greater than 1. The eventual growth rate of each age class is 1.2, which is 20% per year.

To find the eventual relative population sizes, solve $(A - 1.2I)\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} -1.2 & 1.6 & 0 \\ .3 & -.4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad \begin{array}{l} x_1 = (4/3)x_2 \\ x_2 \text{ is free} \end{array}. \text{ Set } \mathbf{v}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

Eventually, there will be about 4 juveniles for every 3 adults.

**c. [M]** Suppose that the initial populations are given by $\mathbf{x}_0 = (15, 10)$. The *Study Guide* describes how to generate the trajectory for as many years as desired and then to plot the values for each population. Let $\mathbf{x}_k = (\mathrm{j}_k, \mathrm{a}_k)$. Then we need to plot the sequences $\{\mathrm{j}_k\}, \{\mathrm{a}_k\}, \{\mathrm{j}_k + \mathrm{a}_k\}$, and $\{\mathrm{j}_k/\mathrm{a}_k\}$. Adjacent points in a sequence can be connected with a line segment. When a sequence is plotted, the resulting graph can be captured on the screen and printed (if done on a computer) or copied by hand onto paper (if working with a graphics calculator).

**18. a.** $A = \begin{bmatrix} 0 & 0 & .42 \\ .6 & 0 & 0 \\ 0 & .75 & .95 \end{bmatrix}$

**b.** $\texttt{ev = eig(A)} = \begin{bmatrix} 0.0774 + 0.4063i \\ 0.0774 - 0.4063i \\ 1.1048 \end{bmatrix}$

The long-term growth rate is 1.105, about 10.5 % per year.

$$\mathbf{v} = \texttt{nulbasis(A-ev(3)*eye(3))} = \begin{bmatrix} 0.3801 \\ 0.2064 \\ 1.0000 \end{bmatrix}$$

For each 100 adults, there will be approximately 38 calves and 21 yearlings.

**Note:** The MATLAB box in the *Study Guide* and the various technology appendices all give directions for generating the sequence of points in a trajectory of a dynamical system. Details for producing a graphical representation of a trajectory are also given, with several options available in MATLAB, Maple, and Mathematica.

## 5.7 SOLUTIONS

**1.** From the "eigendata" (eigenvalues and corresponding eigenvectors) given, the eigenfunctions for the differential equation $\mathbf{x}' = A\mathbf{x}$ are $\mathbf{v}_1 e^{4t}$ and $\mathbf{v}_2 e^{2t}$. The general solution of $\mathbf{x}' = A\mathbf{x}$ has the form

$$c_1\begin{bmatrix} -3 \\ 1 \end{bmatrix}e^{4t} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{2t}$$

The initial condition $\mathbf{x}(0) = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$ determines $c_1$ and $c_2$:

$$c_1\begin{bmatrix} -3 \\ 1 \end{bmatrix}e^{4(0)} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{2(0)} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -1 & -6 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5/2 \\ 0 & 1 & -3/2 \end{bmatrix}$$

Thus $c_1 = 5/2, c_2 = -3/2$, and $\mathbf{x}(t) = \frac{5}{2}\begin{bmatrix} -3 \\ 1 \end{bmatrix}e^{4t} - \frac{3}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{2t}$.

2. From the eigendata given, the eigenfunctions for the differential equation $\mathbf{x}' = A\mathbf{x}$ are $\mathbf{v}_1 e^{-3t}$ and $\mathbf{v}_2 e^{-1t}$. The general solution of $\mathbf{x}' = A\mathbf{x}$ has the form

$$c_1\begin{bmatrix}-1\\1\end{bmatrix}e^{-3t} + c_2\begin{bmatrix}1\\1\end{bmatrix}e^{-1t}$$

The initial condition $\mathbf{x}(0) = \begin{bmatrix}2\\3\end{bmatrix}$ determines $c_1$ and $c_2$:

$$c_1\begin{bmatrix}-1\\1\end{bmatrix}e^{-3(0)} + c_2\begin{bmatrix}1\\1\end{bmatrix}e^{-1(0)} = \begin{bmatrix}2\\3\end{bmatrix}$$

$$\begin{bmatrix}-1 & 1 & 2\\1 & 1 & 3\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1/2\\0 & 1 & 5/2\end{bmatrix}$$

Thus $c_1 = 1/2, c_2 = 5/2$, and $\mathbf{x}(t) = \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix}e^{-3t} + \frac{5}{2}\begin{bmatrix}1\\1\end{bmatrix}e^{-t}$.

3. $A = \begin{bmatrix}2 & 3\\-1 & -2\end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) = 0$. Eigenvalues: 1 and −1.

<u>For $\lambda = 1$:</u> $\begin{bmatrix}1 & 3 & 0\\-1 & -3 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 3 & 0\\0 & 0 & 0\end{bmatrix}$, so $x_1 = -3x_2$ with $x_2$ free. Take $x_2 = 1$ and $\mathbf{v}_1 = \begin{bmatrix}-3\\1\end{bmatrix}$.

<u>For $\lambda = -1$:</u> $\begin{bmatrix}3 & 3 & 0\\-1 & -1 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 0\\0 & 0 & 0\end{bmatrix}$, so $x_1 = -x_2$ with $x_2$ free. Take $x_2 = 1$ and $\mathbf{v}_2 = \begin{bmatrix}-1\\1\end{bmatrix}$.

For the initial condition $\mathbf{x}(0) = \begin{bmatrix}3\\2\end{bmatrix}$, find $c_1$ and $c_2$ such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}(0)$:

$$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{x}(0)\end{bmatrix} = \begin{bmatrix}-3 & -1 & 3\\1 & 1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -5/2\\0 & 1 & 9/2\end{bmatrix}$$

Thus $c_1 = -5/2, c_2 = 9/2$, and $\mathbf{x}(t) = -\frac{5}{2}\begin{bmatrix}-3\\1\end{bmatrix}e^{t} + \frac{9}{2}\begin{bmatrix}-1\\1\end{bmatrix}e^{-t}$.

Since one eigenvalue is positive and the other is negative, the origin is a saddle point of the dynamical system described by $\mathbf{x}' = A\mathbf{x}$. The direction of greatest attraction is the line through $\mathbf{v}_2$ and the origin. The direction of greatest repulsion is the line through $\mathbf{v}_1$ and the origin.

4. $A = \begin{bmatrix}-2 & -5\\1 & 4\end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3) = 0$. Eigenvalues: −1 and 3.

<u>For $\lambda = 3$:</u> $\begin{bmatrix}-5 & -5 & 0\\1 & 1 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 0\\0 & 0 & 0\end{bmatrix}$, so $x_1 = -x_2$ with $x_2$ free. Take $x_2 = 1$ and $\mathbf{v}_1 = \begin{bmatrix}-1\\1\end{bmatrix}$.

<u>For $\lambda = -1$:</u> $\begin{bmatrix}-1 & -5 & 0\\1 & 5 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 5 & 0\\0 & 0 & 0\end{bmatrix}$, so $x_1 = -5x_2$ with $x_2$ free. Take $x_2 = 1$ and $\mathbf{v}_2 = \begin{bmatrix}-5\\1\end{bmatrix}$.

For the initial condition $\mathbf{x}(0) = \begin{bmatrix}3\\2\end{bmatrix}$, find $c_1$ and $c_2$ such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}(0)$:

$$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{x}(0)\end{bmatrix} = \begin{bmatrix}-1 & -5 & 3\\1 & 1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 13/4\\0 & 1 & -5/4\end{bmatrix}$$

Thus $c_1 = 13/4, c_2 = -5/4$, and $\mathbf{x}(t) = \frac{13}{4}\begin{bmatrix}-1\\1\end{bmatrix}e^{3t} - \frac{5}{4}\begin{bmatrix}-5\\1\end{bmatrix}e^{-t}$.

Since one eigenvalue is positive and the other is negative, the origin is a saddle point of the dynamical system described by $\mathbf{x}' = A\mathbf{x}$. The direction of greatest attraction is the line through $\mathbf{v}_2$ and the origin. The direction of greatest repulsion is the line through $\mathbf{v}_1$ and the origin.

5. $A = \begin{bmatrix}7 & -1\\3 & 3\end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 - 10\lambda + 24 = (\lambda - 4)(\lambda - 6) = 0$. Eigenvalues: 4 and 6.

<u>For $\lambda = 4$:</u> $\begin{bmatrix}3 & -1 & 0\\3 & -1 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -1/3 & 0\\0 & 0 & 0\end{bmatrix}$, so $x_1 = (1/3)x_2$ with $x_2$ free. Take $x_2 = 3$ and $\mathbf{v}_1 = \begin{bmatrix}1\\3\end{bmatrix}$.

<u>For $\lambda = 6$:</u> $\begin{bmatrix}1 & -1 & 0\\3 & -3 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -1 & 0\\0 & 0 & 0\end{bmatrix}$, so $x_1 = x_2$ with $x_2$ free. Take $x_2 = 1$ and $\mathbf{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}$.

For the initial condition $\mathbf{x}(0) = \begin{bmatrix}3\\2\end{bmatrix}$, find $c_1$ and $c_2$ such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}(0)$:

$$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{x}(0)\end{bmatrix} = \begin{bmatrix}1 & 1 & 3\\3 & 1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1/2\\0 & 1 & 7/2\end{bmatrix}$$

Thus $c_1 = -1/2, c_2 = 7/2$, and $\mathbf{x}(t) = -\frac{1}{2}\begin{bmatrix}1\\3\end{bmatrix}e^{4t} + \frac{7}{2}\begin{bmatrix}1\\1\end{bmatrix}e^{6t}$.

Since both eigenvalues are positive, the origin is a repellor of the dynamical system described by $\mathbf{x}' = A\mathbf{x}$. The direction of greatest repulsion is the line through $\mathbf{v}_2$ and the origin.

6. $A = \begin{bmatrix}1 & -2\\3 & -4\end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$. Eigenvalues: −1 and −2.

<u>For $\lambda = -2$:</u> $\begin{bmatrix}3 & -2 & 0\\3 & -2 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -2/3 & 0\\0 & 0 & 0\end{bmatrix}$, so $x_1 = (2/3)x_2$ with $x_2$ free. Take $x_2 = 3$ and $\mathbf{v}_1 = \begin{bmatrix}2\\3\end{bmatrix}$.

<u>For $\lambda = -1$:</u> $\begin{bmatrix}2 & -2 & 0\\3 & -3 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -1 & 0\\0 & 0 & 0\end{bmatrix}$, so $x_1 = x_2$ with $x_2$ free. Take $x_2 = 1$ and $\mathbf{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}$.

For the initial condition $\mathbf{x}(0) = \begin{bmatrix}3\\2\end{bmatrix}$, find $c_1$ and $c_2$ such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}(0)$:

$$[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{x}(0)] = \begin{bmatrix}2 & 1 & 3\\3 & 1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1\\0 & 1 & 5\end{bmatrix}$$

Thus $c_1 = -1, c_2 = 5,$ and $\mathbf{x}(t) = -\begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t} + 5\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}.$

Since both eigenvalues are negative, the origin is an attractor of the dynamical system described by $\mathbf{x}' = A\mathbf{x}$. The direction of greatest attraction is the line through $\mathbf{v}_1$ and the origin.

**7.** From Exercise 5, $A = \begin{bmatrix} 7 & -1 \\ 3 & 3 \end{bmatrix}$, with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to eigenvalues 4 and 6 respectively. To decouple the equation $\mathbf{x}' = A\mathbf{x}$, set $P = [\mathbf{v}_1 \ \ \mathbf{v}_2] = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$ and let $D = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$, so that $A = PDP^{-1}$ and $D = P^{-1}AP$. Substituting $\mathbf{x}(t) = P\mathbf{y}(t)$ into $\mathbf{x}' = A\mathbf{x}$ we have

$$\frac{d}{dt}(P\mathbf{y}) = A(P\mathbf{y}) = PDP^{-1}(P\mathbf{y}) = PD\mathbf{y}$$

Since $P$ has constant entries, $\frac{d}{dt}(P\mathbf{y}) = P(\frac{d}{dt}(\mathbf{y}))$, so that left-multiplying the equality $P(\frac{d}{dt}(\mathbf{y})) = PD\mathbf{y}$ by $P^{-1}$ yields $\mathbf{y}' = D\mathbf{y}$, or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

**8.** From Exercise 6, $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$, with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to eigenvalues $-2$ and $-1$ respectively. To decouple the equation $\mathbf{x}' = A\mathbf{x}$, set $P = [\mathbf{v}_1 \ \ \mathbf{v}_2] = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ and let $D = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$, so that $A = PDP^{-1}$ and $D = P^{-1}AP$. Substituting $\mathbf{x}(t) = P\mathbf{y}(t)$ into $\mathbf{x}' = A\mathbf{x}$ we have

$$\frac{d}{dt}(P\mathbf{y}) = A(P\mathbf{y}) = PDP^{-1}(P\mathbf{y}) = PD\mathbf{y}$$

Since $P$ has constant entries, $\frac{d}{dt}(P\mathbf{y}) = P\left(\frac{d}{dt}(\mathbf{y})\right)$, so that left-multiplying the equality $P\left(\frac{d}{dt}(\mathbf{y})\right) = PD\mathbf{y}$ by $P^{-1}$ yields $\mathbf{y}' = D\mathbf{y}$, or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

**9.** $A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$. An eigenvalue of $A$ is $-2+i$ with corresponding eigenvector $\mathbf{v} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$. The complex eigenfunctions $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\overline{\lambda}t}$ form a basis for the set of all complex solutions to $\mathbf{x}' = A\mathbf{x}$. The general complex solution is

$$c_1\begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{(-2+i)t} + c_2\begin{bmatrix} 1+i \\ 1 \end{bmatrix} e^{(-2-i)t}$$

where $c_1$ and $c_2$ are arbitrary complex numbers. To build the general real solution, rewrite $\mathbf{v}e^{(-2+i)t}$ as:

$$\mathbf{v}e^{(-2+i)t} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{-2t}e^{it} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{-2t}(\cos t + i\sin t)$$
$$= \begin{bmatrix} \cos t - i\cos t + i\sin t - i^2\sin t \\ \cos t + i\sin t \end{bmatrix} e^{-2t}$$
$$= \begin{bmatrix} \cos t + \sin t \\ \cos t \end{bmatrix} e^{-2t} + i\begin{bmatrix} \sin t - \cos t \\ \sin t \end{bmatrix} e^{-2t}$$

The general real solution has the form

$$c_1\begin{bmatrix} \cos t + \sin t \\ \cos t \end{bmatrix} e^{-2t} + c_2\begin{bmatrix} \sin t - \cos t \\ \sin t \end{bmatrix} e^{-2t}$$

where $c_1$ and $c_2$ now are real numbers. The trajectories are spirals because the eigenvalues are complex. The spirals tend toward the origin because the real parts of the eigenvalues are negative.

**10.** $A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$. An eigenvalue of $A$ is $2+i$ with corresponding eigenvector $\mathbf{v} = \begin{bmatrix} 1+i \\ -2 \end{bmatrix}$. The complex eigenfunctions $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\overline{\lambda}t}$ form a basis for the set of all complex solutions to $\mathbf{x}' = A\mathbf{x}$. The general complex solution is

$$c_1\begin{bmatrix} 1+i \\ -2 \end{bmatrix} e^{(2+i)t} + c_2\begin{bmatrix} 1-i \\ -2 \end{bmatrix} e^{(2-i)t}$$

where $c_1$ and $c_2$ are arbitrary complex numbers. To build the general real solution, rewrite $\mathbf{v}e^{(2+i)t}$ as:

$$\mathbf{v}e^{(2+i)t} = \begin{bmatrix} 1+i \\ -2 \end{bmatrix} e^{2t}e^{it} = \begin{bmatrix} 1+i \\ -2 \end{bmatrix} e^{2t}(\cos t + i\sin t)$$
$$= \begin{bmatrix} \cos t + i\cos t + i\sin t + i^2\sin t \\ -2\cos t - 2i\sin t \end{bmatrix} e^{2t}$$
$$= \begin{bmatrix} \cos t - \sin t \\ -2\cos t \end{bmatrix} e^{2t} + i\begin{bmatrix} \sin t + \cos t \\ -2\sin t \end{bmatrix} e^{2t}$$

The general real solution has the form

$$c_1\begin{bmatrix} \cos t - \sin t \\ -2\cos t \end{bmatrix} e^{2t} + c_2\begin{bmatrix} \sin t + \cos t \\ -2\sin t \end{bmatrix} e^{2t}$$

where $c_1$ and $c_2$ now are real numbers. The trajectories are spirals because the eigenvalues are complex. The spirals tend away from the origin because the real parts of the eigenvalues are positive.

**11.** $A = \begin{bmatrix} -3 & -9 \\ 2 & 3 \end{bmatrix}$. An eigenvalue of $A$ is $3i$ with corresponding eigenvector $\mathbf{v} = \begin{bmatrix} -3+3i \\ 2 \end{bmatrix}$. The complex eigenfunctions $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\overline{\lambda}t}$ form a basis for the set of all complex solutions to $\mathbf{x}' = A\mathbf{x}$. The general complex solution is

$$c_1\begin{bmatrix}-3+3i\\2\end{bmatrix}e^{(3i)t}+c_2\begin{bmatrix}-3-3i\\2\end{bmatrix}e^{(-3i)t}$$

where $c_1$ and $c_2$ are arbitrary complex numbers. To build the general real solution, rewrite $\mathbf{v}e^{(3i)t}$ as:

$$\mathbf{v}e^{(3i)t}=\begin{bmatrix}-3+3i\\2\end{bmatrix}(\cos 3t+i\sin 3t)$$
$$=\begin{bmatrix}-3\cos 3t-3\sin 3t\\2\cos 3t\end{bmatrix}+i\begin{bmatrix}-3\sin 3t+3\cos 3t\\2\sin 3t\end{bmatrix}$$

The general real solution has the form

$$c_1\begin{bmatrix}-3\cos 3t-3\sin 3t\\2\cos 3t\end{bmatrix}+c_2\begin{bmatrix}-3\sin 3t+3\cos 3t\\2\sin 3t\end{bmatrix}$$

where $c_1$ and $c_2$ now are real numbers. The trajectories are ellipses about the origin because the real parts of the eigenvalues are zero.

12. $A=\begin{bmatrix}-7&10\\-4&5\end{bmatrix}$. An eigenvalue of $A$ is $-1+2i$ with corresponding eigenvector $\mathbf{v}=\begin{bmatrix}3-i\\2\end{bmatrix}$. The complex eigenfunctions $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\overline{\lambda}t}$ form a basis for the set of all complex solutions to $\mathbf{x}'=A\mathbf{x}$. The general complex solution is

$$c_1\begin{bmatrix}3-i\\2\end{bmatrix}e^{(-1+2i)t}+c_2\begin{bmatrix}3+i\\2\end{bmatrix}e^{(-1-2i)t}$$

where $c_1$ and $c_2$ are arbitrary complex numbers. To build the general real solution, rewrite $\mathbf{v}e^{(-1+2i)t}$ as:

$$\mathbf{v}e^{(-1+2i)t}=\begin{bmatrix}3-i\\2\end{bmatrix}e^{-t}(\cos 2t+i\sin 2t)$$
$$=\begin{bmatrix}3\cos 2t+\sin 2t\\2\cos 2t\end{bmatrix}e^{-t}+i\begin{bmatrix}3\sin 2t-\cos 2t\\2\sin 2t\end{bmatrix}e^{-t}$$

The general real solution has the form

$$c_1\begin{bmatrix}3\cos 2t+\sin 2t\\2\cos 2t\end{bmatrix}e^{-t}+c_2\begin{bmatrix}3\sin 2t-\cos 2t\\2\sin 2t\end{bmatrix}e^{-t}$$

where $c_1$ and $c_2$ now are real numbers. The trajectories are spirals because the eigenvalues are complex. The spirals tend toward the origin because the real parts of the eigenvalues are negative.

13. $A=\begin{bmatrix}4&-3\\6&-2\end{bmatrix}$. An eigenvalue of $A$ is $1+3i$ with corresponding eigenvector $\mathbf{v}=\begin{bmatrix}1+i\\2\end{bmatrix}$. The complex eigenfunctions $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\overline{\lambda}t}$ form a basis for the set of all complex solutions to $\mathbf{x}'=A\mathbf{x}$. The general complex solution is

$$c_1\begin{bmatrix}1+i\\2\end{bmatrix}e^{(1+3i)t}+c_2\begin{bmatrix}1-i\\2\end{bmatrix}e^{(1-3i)t}$$

where $c_1$ and $c_2$ are arbitrary complex numbers. To build the general real solution, rewrite $\mathbf{v}e^{(1+3i)t}$ as:

$$\mathbf{v}e^{(1+3i)t}=\begin{bmatrix}1+i\\2\end{bmatrix}e^{t}(\cos 3t+i\sin 3t)$$
$$=\begin{bmatrix}\cos 3t-\sin 3t\\2\cos 3t\end{bmatrix}e^{t}+i\begin{bmatrix}\sin 3t+\cos 3t\\2\sin 3t\end{bmatrix}e^{t}$$

The general real solution has the form

$$c_1\begin{bmatrix}\cos 3t-\sin 3t\\2\cos 3t\end{bmatrix}e^{t}+c_2\begin{bmatrix}\sin 3t+\cos 3t\\2\sin 3t\end{bmatrix}e^{t}$$

where $c_1$ and $c_2$ now are real numbers. The trajectories are spirals because the eigenvalues are complex. The spirals tend away from the origin because the real parts of the eigenvalues are positive.

14. $A=\begin{bmatrix}-2&1\\-8&2\end{bmatrix}$. An eigenvalue of $A$ is $2i$ with corresponding eigenvector $\mathbf{v}=\begin{bmatrix}1-i\\4\end{bmatrix}$. The complex eigenfunctions $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\overline{\lambda}t}$ form a basis for the set of all complex solutions to $\mathbf{x}'=A\mathbf{x}$. The general complex solution is

$$c_1\begin{bmatrix}1-i\\4\end{bmatrix}e^{(2i)t}+c_2\begin{bmatrix}1+i\\4\end{bmatrix}e^{(-2i)t}$$

where $c_1$ and $c_2$ are arbitrary complex numbers. To build the general real solution, rewrite $\mathbf{v}e^{(2i)t}$ as:

$$\mathbf{v}e^{(2i)t}=\begin{bmatrix}1-i\\4\end{bmatrix}(\cos 2t+i\sin 2t)$$
$$=\begin{bmatrix}\cos 2t+\sin 2t\\4\cos 2t\end{bmatrix}+i\begin{bmatrix}\sin 2t-\cos 2t\\4\sin 2t\end{bmatrix}$$

The general real solution has the form

$$c_1\begin{bmatrix}\cos 2t+\sin 2t\\4\cos 2t\end{bmatrix}+c_2\begin{bmatrix}\sin 2t-\cos 2t\\4\sin 2t\end{bmatrix}$$

where $c_1$ and $c_2$ now are real numbers. The trajectories are ellipses about the origin because the real parts of the eigenvalues are zero.

15. **[M]** $A=\begin{bmatrix}-8&-12&-6\\2&1&2\\7&12&5\end{bmatrix}$. The eigenvalues of $A$ are:

```
ev = eig(A)=
 1.0000
-1.0000
-2.0000
nulbasis(A-ev(1)*eye(3)) =
```

```
-1.0000
 0.2500
 1.0000
```

so that $\mathbf{v}_1 = \begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix}$

```
nulbasis(A-ev(2)*eye(3)) =
-1.2000
 0.2000
 1.0000
```

so that $\mathbf{v}_2 = \begin{bmatrix} -6 \\ 1 \\ 5 \end{bmatrix}$

```
nulbasis (A-ev(3)*eye(3)) =
-1.0000
 0.0000
 1.0000
```

so that $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Hence the general solution is $\mathbf{x}(t) = c_1\begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix}e^{t} + c_2\begin{bmatrix} -6 \\ 1 \\ 5 \end{bmatrix}e^{-t} + c_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}e^{-2t}$. The origin is a saddle point.

A solution with $c_1 = 0$ is attracted to the origin while a solution with $c_2 = c_3 = 0$ is repelled.

**16.** **[M]** $A = \begin{bmatrix} -6 & -11 & 16 \\ 2 & 5 & -4 \\ -4 & -5 & 10 \end{bmatrix}$. The eigenvalues of $A$ are:

```
ev = eig(A)=
4.0000
3.0000
2.0000
nulbasis(A-ev(1)*eye(3)) =
 2.3333
-0.6667
 1.0000
```

so that $\mathbf{v}_1 = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}$

```
nulbasis(A-ev(2)*eye(3)) =
 3.0000
-1.0000
 1.0000
```

so that $\mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

```
nulbasis(A-ev(3)*eye(3)) =
2.0000
0.0000
1.0000
```

so that $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Hence the general solution is $\mathbf{x}(t) = c_1\begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}e^{4t} + c_2\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}e^{3t} + c_3\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}e^{2t}$. The origin is a repellor, because all eigenvalues are positive. All trajectories tend away from the origin.

17. **[M]** $A=\begin{bmatrix} 30 & 64 & 23 \\ -11 & -23 & -9 \\ 6 & 15 & 4 \end{bmatrix}$. The eigenvalues of $A$ are:

```
ev = eig(A)=
5.0000 + 2.0000i
5.0000 - 2.0000i
1.0000
nulbasis(A-ev(1)*eye(3))=
7.6667 - 11.3333i
-3.0000 + 4.6667i
1.0000
```

so that $\mathbf{v}_1=\begin{bmatrix} 23-34i \\ -9+14i \\ 3 \end{bmatrix}$

```
nulbasis (A-ev(2)*eye(3))=
7.6667 + 11.3333i
-3.0000 - 4.6667i
1.0000
```

so that $\mathbf{v}_2=\begin{bmatrix} 23+34i \\ -9-14i \\ 3 \end{bmatrix}$

```
nulbasis (A-ev(3)*eye(3))=
-3.0000
1.0000
1.0000
```

so that $\mathbf{v}_3=\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

Hence the general complex solution is

$$\mathbf{x}(t)=c_1\begin{bmatrix} 23-34i \\ -9+14i \\ 3 \end{bmatrix}e^{(5+2i)t}+c_2\begin{bmatrix} 23+34i \\ -9-14i \\ 3 \end{bmatrix}e^{(5-2i)t}+c_3\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}e^{t}$$

Rewriting the first eigenfunction yields

$$\begin{bmatrix} 23-34i \\ -9+14i \\ 3 \end{bmatrix}e^{5t}(\cos 2t+i\sin 2t)=\begin{bmatrix} 23\cos 2t+34\sin 2t \\ -9\cos 2t-14\sin 2t \\ 3\cos 2t \end{bmatrix}e^{5t}+i\begin{bmatrix} 23\sin 2t-34\cos 2t \\ -9\sin 2t+14\cos 2t \\ 3\sin 2t \end{bmatrix}e^{5t}$$

Hence the general real solution is

$$\mathbf{x}(t)=c_1\begin{bmatrix} 23\cos 2t+34\sin 2t \\ -9\cos 2t-14\sin 2t \\ 3\cos 2t \end{bmatrix}e^{5t}+c_2\begin{bmatrix} 23\sin 2t-34\cos 2t \\ -9\sin 2t+14\cos 2t \\ 3\sin 2t \end{bmatrix}e^{5t}+c_3\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}e^{t}$$

where $c_1, c_2,$ and $c_3$ are real. The origin is a repellor, because the real parts of all eigenvalues are positive. All trajectories spiral away from the origin.

18. **[M]** $A=\begin{bmatrix} 53 & -30 & -2 \\ 90 & -52 & -3 \\ 20 & -10 & 2 \end{bmatrix}$. The eigenvalues of $A$ are:

```
ev = eig(A)=
-7.0000
 5.0000 + 1.0000i
 5.0000 - 1.0000i
nulbasis(A-ev(1)*eye(3))=
0.5000
1.0000
0.0000
```

so that $\mathbf{v}_1=\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

```
nulbasis(A-ev(2)*eye(3))=
0.6000 + 0.2000i
0.9000 + 0.3000i
1.0000
```

so that $\mathbf{v}_2=\begin{bmatrix} 6+2i \\ 9+3i \\ 10 \end{bmatrix}$

```
nulbasis(A-ev(3)*eye(3))=
0.6000 - 0.2000i
0.9000 - 0.3000i
1.0000
```

so that $\mathbf{v}_3=\begin{bmatrix} 6-2i \\ 9-3i \\ 10 \end{bmatrix}$

Hence the general complex solution is

$$\mathbf{x}(t)=c_1\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}e^{-7t}+c_2\begin{bmatrix} 6+2i \\ 9+3i \\ 10 \end{bmatrix}e^{(5+i)t}+c_3\begin{bmatrix} 6-2i \\ 9-3i \\ 10 \end{bmatrix}e^{(5-i)t}$$

Rewriting the second eigenfunction yields

$$\begin{bmatrix} 6+2i \\ 9+3i \\ 10 \end{bmatrix} e^{5t}(\cos t + i\sin t) = \begin{bmatrix} 6\cos t - 2\sin t \\ 9\cos t - 3\sin t \\ 10\cos t \end{bmatrix} e^{5t} + i \begin{bmatrix} 6\sin t + 2\cos t \\ 9\sin t + 3\cos t \\ 10\sin t \end{bmatrix} e^{5t}$$

Hence the general real solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} e^{-7t} + c_2 \begin{bmatrix} 6\cos t - 2\sin t \\ 9\cos t - 3\sin t \\ 10\cos t \end{bmatrix} e^{5t} + c_3 \begin{bmatrix} 6\sin t + 2\cos t \\ 9\sin t + 3\cos t \\ 10\sin t \end{bmatrix} e^{5t}$$

where $c_1, c_2,$ and $c_3$ are real. When $c_2 = c_3 = 0$ the trajectories tend toward the origin, and in other cases the trajectories spiral away from the origin.

**19.** **[M]** Substitute $R_1 = 1/5, R_2 = 1/3, C_1 = 4,$ and $C_2 = 3$ into the formula for $A$ given in Example 1, and use a matrix program to find the eigenvalues and eigenvectors:

$$A = \begin{bmatrix} -2 & 3/4 \\ 1 & -1 \end{bmatrix}, \quad \lambda_1 = -.5 : \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \lambda_2 = -2.5 : \mathbf{v}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

The general solution is thus $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-.5t} + c_2 \begin{bmatrix} -3 \\ 2 \end{bmatrix} e^{-2.5t}$. The condition $\mathbf{x}(0) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ implies that $\begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. By a matrix program, $c_1 = 5/2$ and $c_2 = -1/2$, so that

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \mathbf{x}(t) = \frac{5}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-.5t} - \frac{1}{2} \begin{bmatrix} -3 \\ 2 \end{bmatrix} e^{-2.5t}$$

**20.** **[M]** Substitute $R_1 = 1/15, R_2 = 1/3, C_1 = 9,$ and $C_2 = 2$ into the formula for $A$ given in Example 1, and use a matrix program to find the eigenvalues and eigenvectors:

$$A = \begin{bmatrix} -2 & 1/3 \\ 3/2 & -3/2 \end{bmatrix}, \quad \lambda_1 = -1 : \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \lambda_2 = -2.5 : \mathbf{v}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

The general solution is thus $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{-2.5t}$. The condition $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ implies that $\begin{bmatrix} 1 & -2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. By a matrix program, $c_1 = 5/3$ and $c_2 = -2/3$, so that

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \mathbf{x}(t) = \frac{5}{3} \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} - \frac{2}{3} \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{-2.5t}$$

**21.** **[M]** $A = \begin{bmatrix} -1 & -8 \\ 5 & -5 \end{bmatrix}$. Using a matrix program we find that an eigenvalue of $A$ is $-3+6i$ with corresponding eigenvector $\mathbf{v} = \begin{bmatrix} 2+6i \\ 5 \end{bmatrix}$. The conjugates of these form the second eigenvalue-eigenvector pair. The general complex solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2+6i \\ 5 \end{bmatrix} e^{(-3+6i)t} + c_2 \begin{bmatrix} 2-6i \\ 5 \end{bmatrix} e^{(-3-6i)t}$$

where $c_1$ and $c_2$ are arbitrary complex numbers. Rewriting the first eigenfunction and taking its real and imaginary parts, we have

$$\mathbf{v}e^{(-3+6i)t} = \begin{bmatrix} 2+6i \\ 5 \end{bmatrix} e^{-3t}(\cos 6t + i\sin 6t)$$
$$= \begin{bmatrix} 2\cos 6t - 6\sin 6t \\ 5\cos 6t \end{bmatrix} e^{-3t} + i \begin{bmatrix} 2\sin 6t + 6\cos 6t \\ 5\sin 6t \end{bmatrix} e^{-3t}$$

The general real solution has the form

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2\cos 6t - 6\sin 6t \\ 5\cos 6t \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 2\sin 6t + 6\cos 6t \\ 5\sin 6t \end{bmatrix} e^{-3t}$$

where $c_1$ and $c_2$ now are real numbers. To satisfy the initial condition $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 15 \end{bmatrix}$, we solve $c_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \end{bmatrix}$ to get $c_1 = 3, c_2 = -1$. We now have

$$\begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} = \mathbf{x}(t) = 3 \begin{bmatrix} 2\cos 6t - 6\sin 6t \\ 5\cos 6t \end{bmatrix} e^{-3t} - \begin{bmatrix} 2\sin 6t + 6\cos 6t \\ 5\sin 6t \end{bmatrix} e^{-3t} = \begin{bmatrix} -20\sin 6t \\ 15\cos 6t - 5\sin 6t \end{bmatrix} e^{-3t}$$

**22.** **[M]** $A = \begin{bmatrix} 0 & 2 \\ -.4 & -.8 \end{bmatrix}$. Using a matrix program we find that an eigenvalue of $A$ is $-.4+.8i$ with corresponding eigenvector $\mathbf{v} = \begin{bmatrix} -1-2i \\ 1 \end{bmatrix}$. The conjugates of these form the second eigenvalue-eigenvector pair. The general complex solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -1-2i \\ 1 \end{bmatrix} e^{(-.4+.8i)t} + c_2 \begin{bmatrix} -1+2i \\ 1 \end{bmatrix} e^{(-.4-.8i)t}$$

where $c_1$ and $c_2$ are arbitrary complex numbers. Rewriting the first eigenfunction and taking its real and imaginary parts, we have

$$\mathbf{v}e^{(-.4+.8i)t} = \begin{bmatrix} -1-2i \\ 1 \end{bmatrix} e^{-.4t}(\cos .8t + i\sin .8t)$$
$$= \begin{bmatrix} -\cos .8t + 2\sin .8t \\ \cos .8t \end{bmatrix} e^{-.4t} + i \begin{bmatrix} -\sin .8t - 2\cos .8t \\ \sin .8t \end{bmatrix} e^{-.4t}$$

The general real solution has the form

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -\cos .8t + 2\sin .8t \\ \cos .8t \end{bmatrix} e^{-.4t} + c_2 \begin{bmatrix} -\sin .8t - 2\cos .8t \\ \sin .8t \end{bmatrix} e^{-.4t}$$