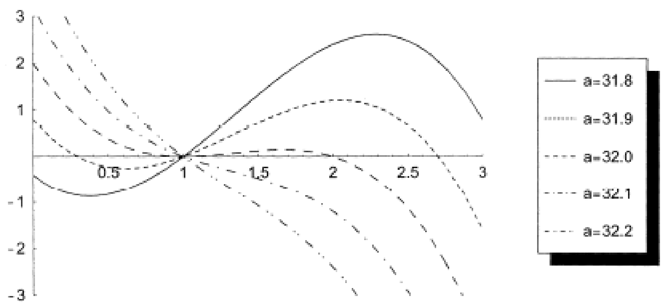

Notes: An appendix in Section 5.3 of the *Study Guide* gives an example of factoring a cubic polynomial with integer coefficients, in case you want your students to find integer eigenvalues of simple 3×3 or perhaps 4×4 matrices.

The MATLAB box for Section 5.3 introduces the command `poly (A)`, which lists the coefficients of the characteristic polynomial of the matrix *A*, and it gives MATLAB code that will produce a graph of the characteristic polynomial. (This is needed for Exercise 30.) The Maple and Mathematica appendices have corresponding information. The appendices for the TI calculators contain only the commands that list the coefficients of the characteristic polynomial.

5.3 SOLUTIONS

1. $P=\begin{bmatrix}5&7\\2&3\end{bmatrix}, D=\begin{bmatrix}2&0\\0&1\end{bmatrix}, A=PDP^{-1}$, and $A^4=PD^4P^{-1}$. We compute

$$P^{-1}=\begin{bmatrix}3&-7\\-2&5\end{bmatrix}, D^4=\begin{bmatrix}16&0\\0&1\end{bmatrix}, \text{ and } A^4=\begin{bmatrix}5&7\\2&3\end{bmatrix}\begin{bmatrix}16&0\\0&1\end{bmatrix}\begin{bmatrix}3&-7\\-2&5\end{bmatrix}=\begin{bmatrix}226&-525\\90&-209\end{bmatrix}$$

2. $P=\begin{bmatrix}1&2\\2&3\end{bmatrix}, D=\begin{bmatrix}1&0\\0&3\end{bmatrix}, A=PDP^{-1}$, and $A^4=PD^4P^{-1}$. We compute

$$P^{-1}=\begin{bmatrix}-3&2\\2&-1\end{bmatrix}, D^4=\begin{bmatrix}1&0\\0&81\end{bmatrix}, \text{ and } A^4=\begin{bmatrix}1&2\\2&3\end{bmatrix}\begin{bmatrix}1&0\\0&81\end{bmatrix}\begin{bmatrix}-3&2\\2&-1\end{bmatrix}=\begin{bmatrix}321&-160\\480&-239\end{bmatrix}$$

3. $A^k=PD^kP^{-1}=\begin{bmatrix}1&0\\2&1\end{bmatrix}\begin{bmatrix}a^k&0\\0&b^k\end{bmatrix}\begin{bmatrix}1&0\\-2&1\end{bmatrix}=\begin{bmatrix}a^k&0\\2a^k-2b^k&b^k\end{bmatrix}.$

4. $A^k=PD^kP^{-1}=\begin{bmatrix}3&-2\\2&-1\end{bmatrix}\begin{bmatrix}(-3)^k&0\\0&(-2)^k\end{bmatrix}\begin{bmatrix}-1&2\\-2&3\end{bmatrix}=\begin{bmatrix}-3\cdot(-3)^k+4\cdot(-2)^k&6\cdot(-3)^k-6\cdot(-2)^k\\-2\cdot(-3)^k+2\cdot(-2)^k&4\cdot(-3)^k-3\cdot(-2)^k\end{bmatrix}.$

5. By the Diagonalization Theorem, eigenvectors form the columns of the left factor, and they correspond respectively to the eigenvalues on the diagonal of the middle factor.

$$\lambda=2:\begin{bmatrix}-1\\1\\-1\end{bmatrix};\lambda=3:\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}0\\-1\\1\end{bmatrix}$$

6. As in Exercise 5, inspection of the factorization gives:

$$\lambda=4:\begin{bmatrix}0\\1\\0\end{bmatrix};\lambda=3:\begin{bmatrix}3\\0\\1\end{bmatrix},\begin{bmatrix}-1\\-3\\0\end{bmatrix}$$

7. Since *A* is triangular, its eigenvalues are obviously ±1.

<u>For $\lambda=1$</u>: $A-1I=\begin{bmatrix}0&0\\6&-2\end{bmatrix}$. The equation $(A-1I)\mathbf{x}=\mathbf{0}$ amounts to $6x_1-2x_2=0$, so $x_1=(1/3)x_2$ with x_2 free. The general solution is $x_2\begin{bmatrix}1/3\\1\end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}1\\3\end{bmatrix}$.

<u>For $\lambda=-1$</u>: $A+1I=\begin{bmatrix}2&0\\6&0\end{bmatrix}$. The equation $(A+1I)\mathbf{x}=\mathbf{0}$ amounts to $2x_1=0$, so $x_1=0$ with x_2 free. The general solution is $x_2\begin{bmatrix}0\\1\end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_2=\begin{bmatrix}0\\1\end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ construct $P=[\mathbf{v}_1\quad\mathbf{v}_2]=\begin{bmatrix}1&0\\3&1\end{bmatrix}$. Then set $D=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$, where the eigenvalues in *D* correspond to $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively.

8. Since *A* is triangular, its only eigenvalue is obviously 3.

<u>For $\lambda=3$</u>: $A-3I=\begin{bmatrix}0&2\\0&0\end{bmatrix}$. The equation $(A-3I)\mathbf{x}=\mathbf{0}$ amounts to $x_2=0$, so $x_2=0$ with x_1 free. The general solution is $x_1\begin{bmatrix}1\\0\end{bmatrix}$. Since we cannot generate an eigenvector basis for $\mathbb{R}^2$, *A* is not diagonalizable.

9. To find the eigenvalues of *A*, compute its characteristic polynomial:

$$\det(A-\lambda I)=\det\begin{bmatrix}2-\lambda&-1\\1&4-\lambda\end{bmatrix}=(2-\lambda)(4-\lambda)-(-1)(1)=\lambda^2-6\lambda+9=(\lambda-3)^2$$

Thus the only eigenvalue of *A* is 3.

<u>For $\lambda=3$</u>: $A-4I=\begin{bmatrix}-1&-1\\1&1\end{bmatrix}$. The equation $(A-3I)\mathbf{x}=\mathbf{0}$ amounts to $x_1+x_2=0$, so $x_1=-x_2$ with x_2 free. The general solution is $x_2\begin{bmatrix}-1\\1\end{bmatrix}$. Since we cannot generate an eigenvector basis for $\mathbb{R}^2$, *A* is not diagonalizable.

10. To find the eigenvalues of A, compute its characteristic polynomial:

$$\det(A-\lambda I)=\det\begin{bmatrix}1-\lambda & 3\\ 4 & 2-\lambda\end{bmatrix}=(1-\lambda)(2-\lambda)-(3)(4)=\lambda^2-3\lambda-10=(\lambda-5)(\lambda+2)$$

Thus the eigenvalues of A are 5 and -2.

For $\lambda=-2$: $A+2I=\begin{bmatrix}3 & 3\\ 4 & 4\end{bmatrix}$. The equation $(A+1I)\mathbf{x}=\mathbf{0}$ amounts to $x_1+x_2=0$, so $x_1=-x_2$ with x_2 free. The general solution is $x_2\begin{bmatrix}-1\\ 1\end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}-1\\ 1\end{bmatrix}$.

For $\lambda=5$: $A-5I=\begin{bmatrix}-4 & 3\\ 4 & -3\end{bmatrix}$. The equation $(A-3I)\mathbf{x}=\mathbf{0}$ amounts to $-4x_1+3x_2=0$, so $x_1=(3/4)x_2$ with x_2 free. The general solution is $x_2\begin{bmatrix}3/4\\ 1\end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_2=\begin{bmatrix}3\\ 4\end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}=\begin{bmatrix}-1 & 3\\ 1 & 4\end{bmatrix}$. Then set $D=\begin{bmatrix}-2 & 0\\ 0 & 5\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively.

11. The eigenvalues of A are given to be -1, and 5.

For $\lambda=-1$: $A+I=\begin{bmatrix}1 & 1 & 1\\ 2 & 2 & 2\\ 3 & 3 & 3\end{bmatrix}$, and row reducing $\begin{bmatrix}A+I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}+x_3\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$, and a nice basis for the eigenspace is $\{\mathbf{v}_1,\mathbf{v}_2\}=\left\{\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix},\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}\right\}$.

For $\lambda=5$: $A-5I=\begin{bmatrix}-5 & 1 & 1\\ 2 & -4 & 2\\ 3 & 3 & -3\end{bmatrix}$, and row reducing $\begin{bmatrix}A-5I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 0 & -1/3 & 0\\ 0 & 1 & -2/3 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}1/3\\ 2/3\\ 1\end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_3=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$.

From $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}=\begin{bmatrix}-1 & -1 & 1\\ 1 & 0 & 2\\ 0 & 1 & 3\end{bmatrix}$. Then set $D=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 5\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively. Note that if changing the order of the vectors in P also changes the order of the diagonal elements in D, and results in the answer given in the text.

12. The eigenvalues of A are given to be 2 and 5.

For $\lambda=2$: $A-2I=\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$, and row reducing $\begin{bmatrix}A-2I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}+x_3\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$, and a nice basis for the eigenspace is $\{\mathbf{v}_1,\mathbf{v}_2\}=\left\{\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix},\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}\right\}$..

For $\lambda=5$: $A-5I=\begin{bmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{bmatrix}$, and row reducing $\begin{bmatrix}A-5I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_3=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$.

From $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}=\begin{bmatrix}-1 & -1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}$. Then set $D=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 5\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

13. The eigenvalues of A are given to be 5 and 1.

For $\lambda=5$: $A-5I=\begin{bmatrix}-3 & 2 & -1\\ 1 & -2 & -1\\ -1 & -2 & -3\end{bmatrix}$, and row reducing $\begin{bmatrix}A-5I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}-1\\ -1\\ 1\end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}-1\\ -1\\ 1\end{bmatrix}$.

For $\lambda=1$: $A-1I=\begin{bmatrix}1 & 2 & -1\\ 1 & 2 & -1\\ -1 & -2 & 1\end{bmatrix}$, and row reducing $\begin{bmatrix}A-I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 2 & -1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}+x_3\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_2,\mathbf{v}_3\}=\left\{\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix},\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}\right\}$.

From $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}=\begin{bmatrix}-1 & -2 & 1\\ -1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$. Then set $D=\begin{bmatrix}5 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

14. The eigenvalues of A are given to be 2 and 3.

For $\lambda = 2$: $A-2I=\begin{bmatrix}0&0&-2\\1&1&2\\0&0&1\end{bmatrix}$, and row reducing $\begin{bmatrix}A-2I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}-1\\1\\0\end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}-1\\1\\0\end{bmatrix}$.

For $\lambda = 3$: $A-3I=\begin{bmatrix}-1&0&-2\\1&0&2\\0&0&0\end{bmatrix}$, and row reducing $\begin{bmatrix}A-3I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&0&2&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}0\\1\\0\end{bmatrix}+x_3\begin{bmatrix}-2\\0\\1\end{bmatrix}$, and a nice basis for the eigenspace is $\{\mathbf{v}_2,\mathbf{v}_3\}=\left\{\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix}\right\}$.

From $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}=\begin{bmatrix}-1&0&-2\\1&1&0\\0&0&1\end{bmatrix}$. Then set $D=\begin{bmatrix}2&0&0\\0&3&0\\0&0&3\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

15. The eigenvalues of A are given to be 0 and 1.

For $\lambda = 0$: $A-0I=\begin{bmatrix}0&-1&-1\\1&2&1\\-1&-1&0\end{bmatrix}$, and row reducing $\begin{bmatrix}A-0I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&0&-1&0\\0&1&1&0\\0&0&0&0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}1\\-1\\1\end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}1\\-1\\1\end{bmatrix}$.

For $\lambda = 1$: $A-I=\begin{bmatrix}-1&-1&-1\\1&1&1\\-1&-1&-1\end{bmatrix}$, and row reducing $\begin{bmatrix}A-I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&1&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}-1\\1\\0\end{bmatrix}+x_3\begin{bmatrix}-1\\0\\1\end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_2,\mathbf{v}_3\}=\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$.

From $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}=\begin{bmatrix}1&-1&-1\\-1&1&0\\1&0&1\end{bmatrix}$. Then set $D=\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively. Note that the answer for P given in the text has the first column scaled by -1, which is also a correct answer, since any nonzero multiple of an eiegnvector is an eigenvector.

16. The only eigenvalue of A given is 0.

For $\lambda = 0$: $A-0I=\begin{bmatrix}1&2&-3\\2&5&-2\\1&3&1\end{bmatrix}$, and row reducing $\begin{bmatrix}A-0I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&0&-11&0\\0&1&4&0\\0&0&0&0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}11\\-4\\1\end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}11\\-4\\1\end{bmatrix}$.

Since $\lambda=0$ has only a one-dimensional eigenspace, we can find at most one linearly independent eigenvector for A, so A is not diagonalizable over the real numbers. The remaining eigenvalues are complex, and this situation is dealt with in Section 5.

17. Since A is triangular, its eigenvalue is obviously 2.

For $\lambda = 2$: $A-2I=\begin{bmatrix}0&0&0\\2&0&0\\2&2&0\end{bmatrix}$, and row reducing $\begin{bmatrix}A-2I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}0\\0\\1\end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}0\\0\\1\end{bmatrix}$.

Since $\lambda=2$ has only a one-dimensional eigenspace, we can find at most one linearly independent eigenvector for A, so A is not diagonalizable.

18. The eigenvalues of A are given to be -2, -1 and 0.

For $\lambda = -2$: $A+2I=\begin{bmatrix}4&-2&-2\\3&-1&-2\\2&-2&0\end{bmatrix}$, and row reducing $\begin{bmatrix}A+2I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&0&-1&0\\0&1&-1&0\\0&0&0&0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}1\\1\\1\end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$.

For $\lambda = -1$: $A+I=\begin{bmatrix}3&-2&-2\\3&-2&-2\\2&-2&-1\end{bmatrix}$, and row reducing $\begin{bmatrix}A+I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&0&-1&0\\0&1&-1/2&0\\0&0&0&0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}1\\1/2\\1\end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_2=\begin{bmatrix}2\\1\\2\end{bmatrix}$.

For $\lambda = 0$: $A-0I=\begin{bmatrix}2&-2&-2\\3&-3&-2\\2&-2&-2\end{bmatrix}$, and row reducing $\begin{bmatrix}A-0I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1&-1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}1\\1\\0\end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_3=\begin{bmatrix}1\\1\\0\end{bmatrix}$.

From $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ construct $P = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$. Then set $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

19. Since A is triangular, its eigenvalues are 2, 3, and 5.

For $\lambda = 2$: $A - 2I = \begin{bmatrix} 3 & -3 & 0 & 9 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and row reducing $[A - 2I \quad \mathbf{0}]$ yields $\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

The general solution is $x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$, and a nice basis for the eigenspace is

$\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

For $\lambda = 3$: $A - 3I = \begin{bmatrix} 2 & -3 & 0 & 9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, and row reducing $[A - 3I \quad \mathbf{0}]$ yields

$\begin{bmatrix} 1 & -3/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2 \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, and a nice basis for the eigenspace is

$\mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} 0 & -3 & 0 & 9 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$, and row reducing $[A - 5I \quad \mathbf{0}]$ yields $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

The general solution is $x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

From $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ construct $P = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4] = \begin{bmatrix} -1 & -1 & 3 & 1 \\ -1 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. Then set

$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ respectively.

Note that this answer differs from the text. There, $P = [\mathbf{v}_4 \ \mathbf{v}_3 \ \mathbf{v}_1 \ \mathbf{v}_2]$ and the entries in D are rearranged to match the new order of the eigenvectors. According to the Diagonalization Theorem, both answers are correct.

20. Since A is triangular, its eigenvalues are 2 and 3.

For $\lambda = 2$: $A - 2I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$, and row reducing $[A - 2I \quad \mathbf{0}]$ yields $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda = 3$: $A - 3I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, and row reducing $[A - 3I \quad \mathbf{0}]$ yields $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

The general solution is $x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Since $\lambda = 3$ has only a one-dimensional eigenspace, we can find at most three linearly independent eigenvectors for A, so A is not diagonalizable.

21. **a.** False. The symbol D does not automatically denote a diagonal matrix.
b. True. See the remark after the statement of the Diagonalization Theorem.
c. False. The 3×3 matrix in Example 4 has 3 eigenvalues, counting multiplicities, but it is not diagonalizable.
d. False. Invertibility depends on 0 not being an eigenvalue. (See the Invertible Matrix Theorem.) A diagonalizable matrix may or may not have 0 as an eigenvalue. See Examples 3 and 5 for both possibilities.

22. **a.** False. The n eigenvectors must be linearly independent. See the Diagonalization Theorem.

b. False. The matrix in Example 3 is diagonalizable, but it has only 2 distinct eigenvalues. (The statement given is the *converse* of Theorem 6.)

c. True. This follows from $AP = PD$ and formulas (1) and (2) in the proof of the Diagonalization Theorem.

d. False. See Example 4. The matrix there is invertible because 0 is not an eigenvalue, but the matrix is not diagonalizable.

23. A is diagonalizable because you know that five linearly independent eigenvectors exist: three in the three-dimensional eigenspace and two in the two-dimensional eigenspace. Theorem 7 guarantees that the set of all five eigenvectors is linearly independent.

24. No, by Theorem 7(b). Here is an explanation that does not appeal to Theorem 7: Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors that span the two one-dimensional eigenspaces. If $\mathbf{v}$ is any other eigenvector, then it belongs to one of the eigenspaces and hence is a multiple of either $\mathbf{v}_1$ or $\mathbf{v}_2$. So there cannot exist three linearly independent eigenvectors. By the Diagonalization Theorem, A cannot be diagonalizable.

25. Let $\{\mathbf{v}_1\}$ be a basis for the one-dimensional eigenspace, let $\mathbf{v}_2$ and $\mathbf{v}_3$ form a basis for the two-dimensional eigenspace, and let $\mathbf{v}_4$ be any eigenvector in the remaining eigenspace. By Theorem 7, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent. Since A is 4×4, the Diagonalization Theorem shows that A is diagonalizable.

26. Yes, if the third eigenspace is only one-dimensional. In this case, the sum of the dimensions of the eigenspaces will be six, whereas the matrix is 7×7. See Theorem 7(b). An argument similar to that for Exercise 24 can also be given.

27. If A is diagonalizable, then $A = PDP^{-1}$ for some invertible P and diagonal D. Since A is invertible, 0 is not an eigenvalue of A. So the diagonal entries in D (which are eigenvalues of A) are not zero, and D is invertible. By the theorem on the inverse of a product,

$$A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$$

Since D^{-1} is obviously diagonal, A^{-1} is diagonalizable.

28. If A has n linearly independent eigenvectors, then by the Diagonalization Theorem, $A = PDP^{-1}$ for some invertible P and diagonal D. Using properties of transposes,

$$\begin{aligned} A^T &= (PDP^{-1})^T = (P^{-1})^T D^T P^T \\ &= (P^T)^{-1} DP^T = QDQ^{-1} \end{aligned}$$

where $Q = (P^T)^{-1}$. Thus A^T is diagonalizable. By the Diagonalization Theorem, the columns of Q are n linearly independent eigenvectors of A^T.

29. The diagonal entries in D_1 are reversed from those in D. So interchange the (eigenvector) columns of P to make them correspond properly to the eigenvalues in D_1. In this case,

$$P_1 = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \text{ and } D_1 = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$

Although the first column of P must be an eigenvector corresponding to the eigenvalue 3, there is nothing to prevent us from selecting some multiple of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, say $\begin{bmatrix} -3 \\ 6 \end{bmatrix}$, and letting $P_2 = \begin{bmatrix} -3 & 1 \\ 6 & -1 \end{bmatrix}$. We now have three different factorizations or "diagonalizations" of A:

$$A = PDP^{-1} = P_1D_1P_1^{-1} = P_2D_1P_2^{-1}$$

30. A nonzero multiple of an eigenvector is another eigenvector. To produce P_2, simply multiply one or both columns of P by a nonzero scalar other than 1.

31. For a 2×2 matrix A to be invertible, its eigenvalues must be nonzero. A first attempt at a construction might be something such as $\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$, whose eigenvalues are 2 and 4. Unfortunately, a 2×2 matrix with two distinct eigenvalues is diagonalizable (Theorem 6). So, adjust the construction to $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$, which works. In fact, any matrix of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ has the desired properties when a and b are nonzero. The eigenspace for the eigenvalue a is one-dimensional, as a simple calculation shows, and there is no other eigenvalue to produce a second eigenvector.

32. Any 2×2 matrix with two distinct eigenvalues is diagonalizable, by Theorem 6. If one of those eigenvalues is zero, then the matrix will not be invertible. Any matrix of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ has the desired properties when a and b are nonzero. The number a must be nonzero to make the matrix diagonalizable; b must be nonzero to make the matrix not diagonal. Other solutions are $\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$ and $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$.

33. $A = \begin{bmatrix} 9 & -4 & -2 & -4 \\ -56 & 32 & -28 & 44 \\ -14 & -14 & 6 & -14 \\ 42 & -33 & 21 & -45 \end{bmatrix}$, `ev = eig(A)=(13,-12,-12, 13)`,

`nulbasis(A-ev(1)*eye(4))` $= \begin{bmatrix} -0.5000 & 0.3333 \\ 0 & -1.3333 \\ 1.0000 & 0 \\ 0 & 1.0000 \end{bmatrix}$.

A basis for the eigenspace of $\lambda = 13$ is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 0 \\ 3 \end{bmatrix} \right\}$.

`nulbasis(A-ev(2)*eye(4))` $\begin{bmatrix} 0.2857 & 0 \\ 1.0000 & -1.0000 \\ 1.0000 & 0 \\ 0 & 1.0000 \end{bmatrix}$,

A basis for the eigenspace of $\lambda = -12$ is $\left\{ \begin{bmatrix} 2 \\ 7 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Thus we construct $P = \begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & -4 & 7 & 1 \\ 2 & 0 & 7 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}$,

and $D = \begin{bmatrix} 13 & 0 & 0 & 0 \\ 0 & 13 & 0 & 0 \\ 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & -12 \end{bmatrix}$. Notice that the anwer in the text lists the eigenvector in an different order in P, and hence the eigenvalues are listed in a different order in D. Both answers are correct.

34. $A = \begin{bmatrix} 4 & -9 & -7 & 8 & 2 \\ -7 & -9 & 0 & 7 & 14 \\ 5 & 10 & 5 & -5 & -10 \\ -2 & 3 & 7 & 0 & 4 \\ -3 & -13 & -7 & 10 & 11 \end{bmatrix}$, `ev = eig(A) = (5,-2,-2,5,5)`,

`nulbasis(A-ev(1)*eye(5))` $= \begin{bmatrix} 2.0000 & -1.0000 & 2.0000 \\ -1.0000 & 1.0000 & 0 \\ 1.0000 & 0 & 0 \\ 0 & 1.0000 & 0 \\ 0 & 0 & 1.0000 \end{bmatrix}$,

A basis for the eigenspace of $\lambda = 5$ is $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

`nulbasis(A-ev(2)*eye(5))` $= \begin{bmatrix} -0.4000 & 0.6000 \\ 1.4000 & 1.4000 \\ -1.0000 & -1.0000 \\ 1.0000 & 0 \\ 0 & 1.0000 \end{bmatrix}$,

A basis for the eigenspace of $\lambda = -2$ is $\left\{ \begin{bmatrix} -2 \\ 7 \\ -5 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -5 \\ 0 \\ 5 \end{bmatrix} \right\}$.

Thus we construct $P = \begin{bmatrix} 2 & -1 & 2 & -2 & 3 \\ -1 & 1 & 0 & 7 & 7 \\ 1 & 0 & 0 & -5 & -5 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$.

35. $A = \begin{bmatrix} 13 & -12 & 9 & -15 & 9 \\ 6 & -5 & 9 & -15 & 9 \\ 6 & -12 & -5 & 6 & 9 \\ 6 & -12 & 9 & -8 & 9 \\ -6 & 12 & 12 & -6 & -2 \end{bmatrix}$, `ev = eig(A) = (7,-14,-14,7,7)`,

`nulbasis(A-ev(1)*eye(5))` $= \begin{bmatrix} 2.0000 & 1.0000 & -1.5000 \\ 1.0000 & 0 & 0 \\ 0 & 1.0000 & 0 \\ 0 & 1.0000 & 0 \\ 0 & 0 & 1.0000 \end{bmatrix}$,

A basis for the eigenspace of $\lambda = 7$ is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\}$.

`nulbasis(A-ev(2)*eye(5))` $= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$,

A basis for the eigenspace of $\lambda = -14$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Thus we construct $P = \begin{bmatrix} 2 & 1 & -3 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} 7 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & -14 & 0 \\ 0 & 0 & 0 & 0 & -14 \end{bmatrix}$.

36. $A = \begin{bmatrix} 24 & -6 & 2 & 6 & 2 \\ 72 & 51 & 9 & -99 & 9 \\ 0 & -63 & 15 & 63 & 63 \\ 72 & 15 & 9 & -63 & 9 \\ 0 & 63 & 21 & -63 & -27 \end{bmatrix}$, `ev = eig(A) = (24,-48,36,-48,36)`,

`nulbasis(A-ev(1)*eye(5))` $= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$,

A basis for the eigenspace of $\lambda = 24$ is $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

`nulbasis(A-ev(2)*eye(5))`,

A basis for the eigenspace of $\lambda = -48$ is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

`nulbasis(A-ev(3)*eye(5))` $= \begin{bmatrix} 1.0000 & -0.3333 \\ 0.0000 & 1.0000 \\ 3.0000 & 0.0000 \\ 1.0000 & 0 \\ 0 & 1.0000 \end{bmatrix}$,

A basis for the eigenspace of $\lambda = 36$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \\ 3 \end{bmatrix} \right\}$.

Thus we construct $P = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & -1 & 3 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 24 & 0 & 0 & 0 & 0 \\ 0 & -48 & 0 & 0 & 0 \\ 0 & 0 & -48 & 0 & 0 \\ 0 & 0 & 0 & 36 & 0 \\ 0 & 0 & 0 & 0 & 36 \end{bmatrix}$. Notice that the anwer in the text lists the eigenvector in an different order in P, and hence the eigenvalues are listed in a different order in D. Both answers are correct.

Notes: For your use, here is another matrix with five distinct real eigenvalues. To four decimal places, they are 11.0654, 9.8785, 3.8238, –3.7332, and –6.0345.

$$\begin{bmatrix} 6 & -8 & 5 & -3 & 0 \\ -7 & 3 & -5 & 3 & 0 \\ -3 & -7 & 5 & -3 & 5 \\ 0 & -4 & 1 & -7 & 5 \\ -5 & -3 & -2 & 0 & 8 \end{bmatrix}$$

The MATLAB box in the *Study Guide* encourages students to use `eig (A)` and `nulbasis` to practice the diagonalization procedure in this section. It also remarks that in later work, a student may automate the process, using the command `[P D]= eig (A)`. You may wish to permit students to use the full power of `eig` in some problems in Sections 5.5 and 5.7.

5.4 SOLUTIONS

1. Since $T(\mathbf{b}_1) = 3\mathbf{d}_1 - 5\mathbf{d}_2, [T(\mathbf{b}_1)]_D = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. Likewise $T(\mathbf{b}_2) = -\mathbf{d}_1 + 6\mathbf{d}_2$ implies that $[T(\mathbf{b}_2)]_D = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$ and $T(\mathbf{b}_3) = 4\mathbf{d}_2$ implies that $[T(\mathbf{b}_3)]_D = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$. Thus the matrix for T relative to B and D is $\left[[T(\mathbf{b}_1)]_D [T(\mathbf{b}_2)]_D [T(\mathbf{b}_3)]_D \right] = \begin{bmatrix} 3 & -1 & 0 \\ -5 & 6 & 4 \end{bmatrix}$.

2. Since $T(\mathbf{d}_1) = 3\mathbf{b}_1 - 3\mathbf{b}_2, [T(\mathbf{d}_1)]_B = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$. Likewise $T(\mathbf{d}_2) = -2\mathbf{b}_1 + 5\mathbf{b}_2$ implies that $[T(\mathbf{d}_2)]_B = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$.

Thus the matrix for T relative to D and B is $\left[[T(\mathbf{d}_1)]_B [T(\mathbf{d}_2)]_B \right] = \begin{bmatrix} 3 & -2 \\ -3 & 5 \end{bmatrix}$.