4.9 SOLUTIONS

Notes: This section builds on the population movement example in Section 1.10. The migration matrix is examined again in Section 5.2, where an eigenvector decomposition shows explicitly why the sequence of state vectors $\mathbf{x}_k$ tends to a steady state vector. The discussion in Section 5.2 does not depend on prior knowledge of this section.

1. a. Let N stand for "News" and M stand for "Music." Then the listeners' behavior is given by the table

From: N	From: M	To:
.7	.6	N
.3	.4	M

so the stochastic matrix is $P = \begin{bmatrix} .7 & .6 \\ .3 & .4 \end{bmatrix}$.

b. Since 100% of the listeners are listening to news at 8: 15, the initial state vector is $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

c. There are two breaks between 8: 15 and 9: 25, so we calculate $\mathbf{x}_2$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .7 & .6 \\ .3 & .4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .7 & .6 \\ .3 & .4 \end{bmatrix}\begin{bmatrix} .7 \\ .3 \end{bmatrix} = \begin{bmatrix} .67 \\ .33 \end{bmatrix}$$

Thus 33% of the listeners are listening to music at 9:25.

2. a. Let the foods be labelled "1," "2," and "3." Then the animals' behavior is given by the table

From: 1	2	3	To:
.6	.2	.2	1
.2	.6	.2	2
.2	.2	.6	3

so the stochastic matrix is $P = \begin{bmatrix} .6 & .2 & .2 \\ .2 & .6 & .2 \\ .2 & .2 & .6 \end{bmatrix}$.

b. There are two trials after the initial trial, so we calculate $\mathbf{x}_2$. The initial state vector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .6 & .2 & .2 \\ .2 & .6 & .2 \\ .2 & .2 & .6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .6 \\ .2 \\ .2 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .6 & .2 & .2 \\ .2 & .6 & .2 \\ .2 & .2 & .6 \end{bmatrix}\begin{bmatrix} .6 \\ .2 \\ .2 \end{bmatrix} = \begin{bmatrix} .44 \\ .28 \\ .28 \end{bmatrix}$$

Thus the probability that the animal will choose food #2 is .28.

3. a. Let H stand for "Healthy" and I stand for "Ill." Then the students' conditions are given by the table

From: H	I	To:
.95	.45	H
.05	.55	I

so the stochastic matrix is $P = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}$.

b. Since 20% of the students are ill on Monday, the initial state vector is $\mathbf{x}_0 = \begin{bmatrix} .8 \\ .2 \end{bmatrix}$. For Tuesday's percentages, we calculate $\mathbf{x}_1$; for Wednesday's percentages, we calculate $\mathbf{x}_2$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}\begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .85 \\ .15 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}\begin{bmatrix} .85 \\ .15 \end{bmatrix} = \begin{bmatrix} .875 \\ .125 \end{bmatrix}$$

Thus 15% of the students are ill on Tuesday, and 12.5% are ill on Wednesday.

c. Since the student is well today, the initial state vector is $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We calculate $\mathbf{x}_2$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .95 \\ .05 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}\begin{bmatrix} .95 \\ .05 \end{bmatrix} = \begin{bmatrix} .925 \\ .075 \end{bmatrix}$$

Thus the probability that the student is well two days from now is .925.

4. a. Let G stand for good weather, I for indifferent weather, and B for bad weather. Then the change in the weather is given by the table

From: G	I	B	To:
.4	.5	.3	G
.3	.2	.4	I
.3	.3	.3	B

so the stochastic matrix is $P = \begin{bmatrix} .4 & .5 & .3 \\ .3 & .2 & .4 \\ .3 & .3 & .3 \end{bmatrix}$.

b. The initial state vector is $\begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix}$. We calculate $\mathbf{x}_1$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .4 & .5 & .3 \\ .3 & .2 & .4 \\ .3 & .3 & .3 \end{bmatrix}\begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix} = \begin{bmatrix} .45 \\ .25 \\ .30 \end{bmatrix}$$

Thus the chance of bad weather tomorrow is 30%.

c. The initial state vector is $\mathbf{x}_0 = \begin{bmatrix} 0 \\ .6 \\ .4 \end{bmatrix}$. We calculate $\mathbf{x}_2$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .4 & .5 & .3 \\ .3 & .2 & .4 \\ .3 & .3 & .3 \end{bmatrix}\begin{bmatrix} 0 \\ .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .42 \\ .28 \\ .30 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .4 & .5 & .3 \\ .3 & .2 & .4 \\ .3 & .3 & .3 \end{bmatrix}\begin{bmatrix} .42 \\ .28 \\ .30 \end{bmatrix} = \begin{bmatrix} .398 \\ .302 \\ .300 \end{bmatrix}$$

Thus the chance of good weather on Wednesday is 39.8%, or approximately 40%.

5. We solve $P\mathbf{x} = \mathbf{x}$ by rewriting the equation as $(P - I)\mathbf{x} = \mathbf{0}$, where $P - I = \begin{bmatrix} -.9 & .5 \\ .9 & -.5 \end{bmatrix}$. Row reducing the augmented matrix for the homogeneous system $(P - I)\mathbf{x} = \mathbf{0}$ gives

$$\begin{bmatrix} -.9 & .5 & 0 \\ .9 & -.5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5/9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 5/9 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 5 \\ 9 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 5 \\ 9 \end{bmatrix}$ sum to 14, multiply by 1/14 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} 5/14 \\ 9/14 \end{bmatrix}$.

6. We solve $P\mathbf{x} = \mathbf{x}$ by rewriting the equation as $(P - I)\mathbf{x} = \mathbf{0}$, where $P - I = \begin{bmatrix} -.6 & .8 \\ .6 & -.8 \end{bmatrix}$. Row reducing the augmented matrix for the homogeneous system $(P - I)\mathbf{x} = \mathbf{0}$ gives

$$\begin{bmatrix} -.6 & .8 & 0 \\ .6 & -.8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 4/3 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ sum to 7, multiply by 1/7 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} 4/7 \\ 3/7 \end{bmatrix} \approx \begin{bmatrix} .571 \\ .429 \end{bmatrix}$.

7. We solve $P\mathbf{x} = \mathbf{x}$ by rewriting the equation as $(P - I)\mathbf{x} = \mathbf{0}$, where $P - I = \begin{bmatrix} -.3 & .1 & .1 \\ .2 & -.2 & .2 \\ .1 & .1 & -.3 \end{bmatrix}$. Row reducing the augmented matrix for the homogeneous system $(P - I)\mathbf{x} = \mathbf{0}$ gives

$$\begin{bmatrix} -.3 & .1 & .1 & 0 \\ .2 & -.2 & .2 & 0 \\ .1 & .1 & -.3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ sum to 4, multiply by 1/4 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix} = \begin{bmatrix} .25 \\ .5 \\ .25 \end{bmatrix}$.

8. We solve $P\mathbf{x} = \mathbf{x}$ by rewriting the equation as $(P - I)\mathbf{x} = \mathbf{0}$, where $P - I = \begin{bmatrix} -.6 & .5 & .8 \\ 0 & -.5 & .1 \\ .6 & 0 & -.9 \end{bmatrix}$. Row reducing the augmented matrix for the homogeneous system $(P - I)\mathbf{x} = \mathbf{0}$ gives

$$\begin{bmatrix} -.6 & .5 & .8 & 0 \\ 0 & -.5 & .1 & 0 \\ .6 & 0 & -.9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3/2 & 0 \\ 0 & 1 & -1/5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 3/2 \\ 1/5 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 15 \\ 2 \\ 10 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 15 \\ 2 \\ 10 \end{bmatrix}$ sum to 27, multiply by 1/27 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} 15/27 \\ 2/27 \\ 10/27 \end{bmatrix} \approx \begin{bmatrix} .556 \\ .074 \\ .370 \end{bmatrix}$.

9. Since $P^2 = \begin{bmatrix} .84 & .2 \\ .16 & .8 \end{bmatrix}$ has all positive entries, P is a regular stochastic matrix.

10. Since $P^k = \begin{bmatrix} 1 & 1-.7^k \\ 0 & .7^k \end{bmatrix}$ will have a zero as its (2,1) entry for all k, P is not a regular stochastic matrix.

11. a. From Exercise 1, $P = \begin{bmatrix} .7 & .6 \\ .3 & .4 \end{bmatrix}$, so $P - I = \begin{bmatrix} -.3 & .6 \\ .3 & -.6 \end{bmatrix}$. Solving $(P - I)\mathbf{x} = \mathbf{0}$ by row reducing the augmented matrix gives

$$\begin{bmatrix}-.3 & .6 & 0\\ .3 & -.6 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Thus $\mathbf{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = x_2\begin{bmatrix}2\\1\end{bmatrix}$, and one solution is $\begin{bmatrix}2\\1\end{bmatrix}$. Since the entries in $\begin{bmatrix}2\\1\end{bmatrix}$ sum to 3, multiply by 1/3 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix}2/3\\1/3\end{bmatrix} \approx \begin{bmatrix}.667\\.333\end{bmatrix}$.

b. Since $\mathbf{q} = \begin{bmatrix}2/3\\1/3\end{bmatrix}$, 2/3 of the listeners will be listening to the news at some time late in the day.

12. From Exercise 2, $P = \begin{bmatrix}.6 & .2 & .2\\ .2 & .6 & .2\\ .2 & .2 & .6\end{bmatrix}$, so $P - I = \begin{bmatrix}-.4 & .2 & .2\\ .2 & -.4 & .2\\ .2 & .2 & -.4\end{bmatrix}$. Solving $(P - I)\mathbf{x} = \mathbf{0}$ by row reducing the augmented matrix gives

$$\begin{bmatrix}-.4 & .2 & .2 & 0\\ .2 & -.4 & .2 & 0\\ .2 & .2 & -.4 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Thus $\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_3\begin{bmatrix}1\\1\\1\end{bmatrix}$, and one solution is $\begin{bmatrix}1\\1\\1\end{bmatrix}$. Since the entries in $\begin{bmatrix}1\\1\\1\end{bmatrix}$ sum to 3, multiply by 1/3 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix}1/3\\1/3\\1/3\end{bmatrix} \approx \begin{bmatrix}.333\\.333\\.333\end{bmatrix}$. Thus in the long run each food will be preferred equally.

13. a. From Exercise 3, $P = \begin{bmatrix}.95 & .45\\ .05 & .55\end{bmatrix}$, so $P - I = \begin{bmatrix}-.05 & .45\\ .05 & -.45\end{bmatrix}$. Solving $(P - I)\mathbf{x} = \mathbf{0}$ by row reducing the augmented matrix gives

$$\begin{bmatrix}-.05 & .45 & 0\\ .05 & -.45 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -9 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Thus $\mathbf{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = x_2\begin{bmatrix}9\\1\end{bmatrix}$, and one solution is $\begin{bmatrix}9\\1\end{bmatrix}$. Since the entries in $\begin{bmatrix}9\\1\end{bmatrix}$ sum to 10, multiply by 1/10 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix}9/10\\1/10\end{bmatrix} = \begin{bmatrix}.9\\.1\end{bmatrix}$.

b. After many days, a specific student is ill with probability .1, and it does not matter whether that student is ill today or not.

14. From Exercise 4, $P = \begin{bmatrix}.4 & .5 & .3\\ .3 & .2 & .4\\ .3 & .3 & .3\end{bmatrix}$, so $P - I = \begin{bmatrix}-.6 & .5 & .3\\ .3 & -.8 & .4\\ .3 & .3 & -.7\end{bmatrix}$. Solving $(P - I)\mathbf{x} = \mathbf{0}$ by row reducing the augmented matrix gives

$$\begin{bmatrix}-.6 & .5 & .3 & 0\\ .3 & -.8 & .4 & 0\\ .3 & .3 & -.7 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -4/3 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Thus $\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_3\begin{bmatrix}4/3\\1\\1\end{bmatrix}$, and one solution is $\begin{bmatrix}4\\3\\3\end{bmatrix}$. Since the entries in $\begin{bmatrix}4\\3\\3\end{bmatrix}$ sum to 10, multiply by 1/10 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix}4/10\\3/10\\3/10\end{bmatrix} = \begin{bmatrix}.4\\.3\\.3\end{bmatrix}$. Thus in the long run the chance that a day has good weather is 40%.

15. [M] Let $P = \begin{bmatrix}.9821 & .0029\\ .0179 & .9971\end{bmatrix}$, so $P - I = \begin{bmatrix}-.0179 & .0029\\ .0179 & -.0029\end{bmatrix}$. Solving $(P - I)\mathbf{x} = \mathbf{0}$ by row reducing the augmented matrix gives

$$\begin{bmatrix}-.0179 & .0029 & 0\\ .0179 & -.0029 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -.162011 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Thus $\mathbf{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = x_2\begin{bmatrix}.162011\\1\end{bmatrix}$, and one solution is $\begin{bmatrix}.162011\\1\end{bmatrix}$. Since the entries in $\begin{bmatrix}.162011\\1\end{bmatrix}$ sum to 1.162011, multiply by 1/1.162011 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix}.139423\\.860577\end{bmatrix}$. Thus about 13.9% of the total U.S. population would eventually live in California.

16. [M] Let $P = \begin{bmatrix}.90 & .01 & .09\\ .01 & .90 & .01\\ .09 & .09 & .90\end{bmatrix}$, so $P - I = \begin{bmatrix}-.10 & .01 & .09\\ .01 & -.10 & .01\\ .09 & .09 & -.1\end{bmatrix}$. Solving $(P - I)\mathbf{x} = \mathbf{0}$ by row reducing the augmented matrix gives

$$\begin{bmatrix}-.10 & .01 & .09 & 0\\ .01 & -.10 & .01 & 0\\ .09 & .09 & -.1 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -.919192 & 0\\ 0 & 1 & -.191919 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Thus $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_3\begin{bmatrix}.919192\\.191919\\1\end{bmatrix}$, and one solution is $\begin{bmatrix}.919192\\.191919\\1\end{bmatrix}$. Since the entries in $\begin{bmatrix}.919192\\.191919\\1\end{bmatrix}$ sum to 2.111111, multiply by 1/2.111111 to obtain the steady-state vector $\mathbf{q}=\begin{bmatrix}.435407\\.090909\\.473684\end{bmatrix}$. Thus on a typical day, about (.090909)(2000) = 182 cars will be rented or available from the downtown location.

17. **a.** The entries in each column of P sum to 1. Each column in the matrix $P-I$ has the same entries as in P except one of the entries is decreased by 1. Thus the entries in each column of $P-I$ sum to 0, and adding all of the other rows of $P-I$ to its bottom row produces a row of zeros.

b. By part a., the bottom row of $P-I$ is the negative of the sum of the other rows, so the rows of $P-I$ are linearly dependent.

c. By part b. and the Spanning Set Theorem, the bottom row of $P-I$ can be removed and the remaining $(n-1)$ rows will still span the row space of $P-I$. Thus the dimension of the row space of $P-I$ is less than n. Alternatively, let A be the matrix obtained from $P-I$ by adding to the bottom row all the other rows. These row operations did not change the row space, so the row space of $P-I$ is spanned by the nonzero rows of A. By part a., the bottom row of A is a zero row, so the row space of $P-I$ is spanned by the first $(n-1)$ rows of A.

d. By part c., the rank of $P-I$ is less than n, so the Rank Theorem may be used to show that $\dim\text{Nul}(P-I)=n-\text{rank}(P-I)>0$. Alternatively the Invertible Martix Theorem may be used since $P-I$ is a square matrix.

18. If $\alpha=\beta=0$ then $P=\begin{bmatrix}1&0\\0&1\end{bmatrix}$. Notice that $P\mathbf{x}=\mathbf{x}$ for any vector $\mathbf{x}$ in $\mathbb{R}^2$, and that $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ are two linearly independent steady-state vectors in this case.

If $\alpha\neq 0$ or $\beta\neq 0$, we solve $(P-I)\mathbf{x}=\mathbf{0}$ where $P-I=\begin{bmatrix}-\alpha&\beta\\\alpha&-\beta\end{bmatrix}$. Row reducing the augmented matrix gives

$$\begin{bmatrix}-\alpha&\beta&0\\\alpha&-\beta&0\end{bmatrix}\sim\begin{bmatrix}\alpha&-\beta&0\\0&0&0\end{bmatrix}$$

So $\alpha x_1=\beta x_2$, and one possible solution is to let $x_1=\beta$, $x_2=\alpha$. Thus $\mathbf{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}\beta\\\alpha\end{bmatrix}$. Since the entries in $\begin{bmatrix}\beta\\\alpha\end{bmatrix}$ sum to $\alpha+\beta$, multiply by $1/(\alpha+\beta)$ to obtain the steady-state vector $\mathbf{q}=\frac{1}{\alpha+\beta}\begin{bmatrix}\beta\\\alpha\end{bmatrix}$.

19. **a.** The product $S\mathbf{x}$ equals the sum of the entries in $\mathbf{x}$. Thus $\mathbf{x}$ is a probability vector if and only if its entries are nonnegative and $S\mathbf{x}=1$.

b. Let $P=[\mathbf{p}_1\quad\mathbf{p}_2\quad\dots\quad\mathbf{p}_n]$, where $\mathbf{p}_1,\mathbf{p}_2,\dots,\mathbf{p}_n$ are probability vectors. By part a.,

$$SP=[S\mathbf{p}_1\quad S\mathbf{p}_2\quad\dots\quad S\mathbf{p}_n]=[1\quad 1\quad\dots\quad 1]=S$$

c. By part b., $S(P\mathbf{x})=(SP)\mathbf{x}=S\mathbf{x}=1$. The entries in $P\mathbf{x}$ are nonnegative since P and $\mathbf{x}$ have only nonnegative entries. By part a., the condition $S(P\mathbf{x})=1$ shows that $P\mathbf{x}$ is a probability vector.

20. Let $P=[\mathbf{p}_1\quad\mathbf{p}_2\quad\dots\quad\mathbf{p}_n]$, so $P^2=PP=[P\mathbf{p}_1\quad P\mathbf{p}_2\quad\dots\quad P\mathbf{p}_n]$. By Exercise 19c., the columns of P^2 are probability vectors, so P^2 is a stochastic matrix.

Alternatively, $SP=S$ by Exercise 19b., since P is a stochastic matrix. Right multiplication by P gives $SP^2=SP$, so $SP=S$ implies that $SP^2=S$. Since the entries in P are nonnegative, so are the entries in P^2, and P^2 is stochastic matrix.

21. **[M]**

a. To four decimal places,

$$P^2=\begin{bmatrix}.2779&.2780&.2803&.2941\\.3368&.3355&.3357&.3335\\.1847&.1861&.1833&.1697\\.2005&.2004&.2007&.2027\end{bmatrix},\ P^3=\begin{bmatrix}.2817&.2817&.2817&.2814\\.3356&.3356&.3355&.3352\\.1817&.1817&.1819&.1825\\.2010&.2010&.2010&.2009\end{bmatrix},$$

$$P^4=P^5=\begin{bmatrix}.2816&.2816&.2816&.2816\\.3355&.3355&.3355&.3355\\.1819&.1819&.1819&.1819\\.2009&.2009&.2009&.2009\end{bmatrix}$$

The columns of P^k are converging to a common vector as k increases. The steady state vector $\mathbf{q}$ for P is $\mathbf{q}=\begin{bmatrix}.2816\\.3355\\.1819\\.2009\end{bmatrix}$, which is the vector to which the columns of P^k are converging.

b. To four decimal places,

$$Q^{10}=\begin{bmatrix}.8222&.4044&.5385\\.0324&.3966&.1666\\.1453&.1990&.2949\end{bmatrix},\ Q^{20}=\begin{bmatrix}.7674&.6000&.6690\\.0637&.2036&.1326\\.1688&.1964&.1984\end{bmatrix},$$

$$Q^{30}=\begin{bmatrix}.7477&.6815&.7105\\.0783&.1329&.1074\\.1740&.1856&.1821\end{bmatrix},\ Q^{40}=\begin{bmatrix}.7401&.7140&.7257\\.0843&.1057&.0960\\.1756&.1802&.1783\end{bmatrix},$$

$$Q^{50}=\begin{bmatrix}.7372&.7269&.7315\\.0867&.0951&.0913\\.1761&.1780&.1772\end{bmatrix},\ Q^{60}=\begin{bmatrix}.7360&.7320&.7338\\.0876&.0909&.0894\\.1763&.1771&.1767\end{bmatrix},$$

$$Q^{70}=\begin{bmatrix}.7356&.7340&.7347\\.0880&.0893&.0887\\.1764&.1767&.1766\end{bmatrix},\ Q^{80}=\begin{bmatrix}.7354&.7348&.7351\\.0881&.0887&.0884\\.1764&.1766&.1765\end{bmatrix},$$

$$Q^{116} = Q^{117} = \begin{bmatrix} .7353 & .7353 & .7353 \\ .0882 & .0882 & .0882 \\ .1765 & .1765 & .1765 \end{bmatrix}$$

The steady state vector $\mathbf{q}$ for Q is $\mathbf{q} = \begin{bmatrix} .7353 \\ .0882 \\ .1765 \end{bmatrix}$ Conjecture: the columns of P^k, where P is a regular stochastic matrix, converge to the steady state vector for P as k increases.

c. Let P be an $n \times n$ regular stochastic matrix, $\mathbf{q}$ the steady state vector of P, and $\mathbf{e}_j$ the j^{th} column of the $n \times n$ identity matrix. Consider the Markov chain $\{\mathbf{x}_k\}$ where $\mathbf{x}_{k+1} = P\mathbf{x}_k$ and $\mathbf{x}_0 = e_j$. By Theorem 18, $\mathbf{x}_k = P^k\mathbf{x}_0$ converges to $\mathbf{q}$ as $k \to \infty$. But $P^k\mathbf{x}_0 = P^k\mathbf{e}_j$, which is the j^{th} column of P^k. Thus the j^{th} column of P^k converges to $\mathbf{q}$ as $k \to \infty$; that is, $P^k \to [\mathbf{q} \quad \mathbf{q} \quad \ldots \quad \mathbf{q}]$.

22. **[M]** Answers will vary.

MATLAB Student Version 4.0 code for Method (1):

```
A=randstoc(32); flops(0);
tic, x=nulbasis(A-eye(32));
q=x/sum(x); toc, flops
```

MATLAB Student Version 4.0 code for Method (2):

```
A=randstoc(32); flops(0);
tic, B=A^100; q=B(: ,1); toc, flops
```

Chapter 4 SUPPLEMENTARY EXERCISES

1. **a.** True. This set is $\text{Span}\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$, and every subspace is itself a vector space.

b. True. Any linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}$ is also a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}, \mathbf{v}_p$ using the zero weight on $\mathbf{v}_p$.

c. False. Counterexample: Take $\mathbf{v}_p = 2\mathbf{v}_1$. Then $\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ is linearly dependent.

d. False. Counterexample: Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$. Then $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a linearly independent set but is not a basis for $\mathbb{R}^3$.

e. True. See the Spanning Set Theorem (Section 4.3).

f. True. By the Basis Theorem, S is a basis for V because S spans V and has exactly p elements. So S must be linearly independent.

g. False. The plane must pass through the origin to be a subspace.

h. False. Counterexample: $\begin{bmatrix} 2 & 5 & -2 & 0 \\ 0 & 0 & 7 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

i. True. This statement appears before Theorem 13 in Section 4.6.

j. False. Row operations on A do not change the solutions of $A\mathbf{x} = \mathbf{0}$.

k. False. Counterexample: $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$; A has two nonzero rows but the rank of A is 1.

l. False. If U has k nonzero rows, then rank $A = k$ and dimNul $A = n - k$ by the Rank Theorem.

m. True. Row equivalent matrices have the same number of pivot columns.

n. False. The nonzero rows of A span Row A but they may not be linearly independent.

o. True. The nonzero rows of the reduced echelon form E form a basis for the row space of each matrix that is row equivalent to E.

p. True. If H is the zero subspace, let A be the 3×3 zero matrix. If dim $H = 1$, let $\{\mathbf{v}\}$ be a basis for H and set $A = [\mathbf{v} \quad \mathbf{v} \quad \mathbf{v}]$. If dim $H = 2$, let $\{\mathbf{u},\mathbf{v}\}$ be a basis for H and set $A = [\mathbf{u} \quad \mathbf{v} \quad \mathbf{v}]$, for example. If dim $H = 3$, then $H = \mathbb{R}^3$, so A can be any 3×3 invertible matrix. Or, let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be a basis for H and set $A = [\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}]$.

q. False. Counterexample: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. If rank $A = n$ (the number of *columns* in A), then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

r. True. If $\mathbf{x} \mapsto A\mathbf{x}$ is onto, then Col $A = \mathbb{R}^m$ and rank $A = m$. See Theorem 12(a) in Section 1.9.

s. True. See the second paragraph after Theorem 15 in Section 4.7.

t. False. The j^{th} column of $\underset{C \leftarrow B}{P}$ is $[\mathbf{b}_j]_C$.

2. The set is Span$S$, where $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -8 \\ 7 \\ 1 \end{bmatrix} \right\}$. Note that S is a linearly dependent set, but each pair of vectors in S forms a linearly independent set. Thus any two of the three vectors $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -8 \\ 7 \\ 1 \end{bmatrix}$ will be a basis for Span$S$.

3. The vector $\mathbf{b}$ will be in $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ if and only if there exist constants c_1 and c_2 with $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \mathbf{b}$. Row reducing the augmented matrix gives

$$\begin{bmatrix} -2 & 1 & b_1 \\ 4 & 2 & b_2 \\ -6 & -5 & b_3 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & b_1 \\ 0 & 4 & 2b_1 + b_2 \\ 0 & 0 & b_1 + 2b_2 + b_3 \end{bmatrix}$$

so $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ is the set of all (b_1, b_2, b_3) satisfying $b_1 + 2b_2 + b_3 = 0$.

4. The vector **g** is not a scalar multiple of the vector **f**, and **f** is not a scalar multiple of **g**, so the set $\{\mathbf{f}, \mathbf{g}\}$ is linearly independent. Even though the *number* $\mathbf{g}(t)$ is a scalar multiple of $\mathbf{f}(t)$ for each t, the scalar depends on t.

5. The vector $\mathbf{p}_1$ is not zero, and $\mathbf{p}_2$ is not a multiple of $\mathbf{p}_1$. However, $\mathbf{p}_3$ is $2\mathbf{p}_1 + 2\mathbf{p}_2$, so $\mathbf{p}_3$ is discarded. The vector $\mathbf{p}_4$ cannot be a linear combination of $\mathbf{p}_1$ and $\mathbf{p}_2$ since $\mathbf{p}_4$ involves t^2 but $\mathbf{p}_1$ and $\mathbf{p}_2$ do not involve t^2. The vector $\mathbf{p}_5$ is $(3/2)\mathbf{p}_1 - (1/2)\mathbf{p}_2 + \mathbf{p}_4$ (which may not be so easy to see at first.) Thus $\mathbf{p}_5$ is a linear combination of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_4$, so $\mathbf{p}_5$ is discarded. So the resulting basis is $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_4\}$.

6. Find two polynomials from the set $\{\mathbf{p}_1, \ldots, \mathbf{p}_4\}$ that are not multiples of one another. This is easy, because one compares only two polynomials at a time. Since these two polynomials form a linearly independent set in a two-dimensional space, they form a basis for H by the Basis Theorem.

7. You would have to know that the solution set of the homogeneous system is spanned by two solutions. In this case, the null space of the 18×20 coefficient matrix A is at most two-dimensional. By the Rank Theorem, $\dim \text{Col } A = 20 - \dim \text{Nul } A \geq 20 - 2 = 18$. Since Col A is a subspace of $\mathbb{R}^{18}$, Col $A = \mathbb{R}^{18}$. Thus $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^{18}$.

8. If $n = 0$, then H and V are both the zero subspace, and $H = V$. If $n > 0$, then a basis for H consists of n linearly independent vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$. These vectors are also linearly independent as elements of V. But since $\dim V = n$, any set of n linearly independent vectors in V must be a basis for V by the Basis Theorem. So $\mathbf{u}_1, \ldots, \mathbf{u}_n$ span V, and $H = \text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\} = V$.

9. Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let A be the $m \times n$ standard matrix of T.

a. If T is one-to-one, then the columns of A are linearly independent by Theorem 12 in Section 1.9, so $\dim \text{Nul } A = 0$. By the Rank Theorem, $\dim \text{Col } A = n - 0 = n$, which is the number of columns of A. As noted in Section 4.2, the range of T is Col A, so the dimension of the range of T is n.

b. If T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$, then the columns of A span $\mathbb{R}^m$ by Theoerm 12 in Section 1.9, so $\dim \text{Col } A = m$. By the Rank Theorem, $\dim \text{Nul } A = n - m$. As noted in Section 4.2, the kernel of T is Nul A, so the dimension of the kernel of T is $n - m$. Note that $n - m$ must be nonnegative in this case: since A must have a pivot in each row, $n \geq m$.

10. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. If S were linearly independent and not a basis for V, then S would not span V. In this case, there would be a vector $\mathbf{v}_{p+1}$ in V that is not in $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Let $S' = \{\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{v}_{p+1}\}$. Then S' is linearly independent since none of the vectors in S' is a linear combination of vectors that precede it. Since S' has more elements than S, this would contradict the maximality of S. Hence S must be a basis for V.

11. If S is a finite spanning set for V, then a subset of S is a basis for V. Denote this subset of S by S'. Since S' is a basis for V, S' must span V. Since S is a minimal spanning set, S' cannot be a proper subset of S. Thus $S' = S$, and S is a basis for V.

12. a. Let **y** be in Col AB. Then $\mathbf{y} = AB\mathbf{x}$ for some **x**. But $AB\mathbf{x} = A(B\mathbf{x})$, so $\mathbf{y} = A(B\mathbf{x})$, and **y** is in Col A. Thus Col AB is a subspace of Col A, so rank $AB = \dim \text{Col } AB \leq \dim \text{Col } A = \text{rank } A$ by Theorem 11 in Section 4.5.

b. By the Rank Theorem and part a.:

$$\text{rank } AB = \text{rank}(AB)^T = \text{rank } B^T A^T \leq \text{rank } B^T = \text{rank } B$$

13. By Exercise 12, rank $PA \leq$ rank A, and $\text{rank } A = \text{rank}(P^{-1}P)A = \text{rank } P^{-1}(PA) \leq \text{rank } PA$, so rank PA = rank A.

14. Note that $(AQ)^T = Q^T A^T$. Since Q^T is invertible, we can use Exercise 13 to conclude that $\text{rank}(AQ)^T = \text{rank } Q^T A^T = \text{rank } A^T$. Since the ranks of a matrix and its transpose are equal (by the Rank Theorem), rank AQ = rank A.

15. The equation $AB = O$ shows that each column of B is in Nul A. Since Nul A is a subspace of $\mathbb{R}^n$, all linear combinations of the columns of B are in Nul A. That is, Col B is a subspace of Nul A. By Theorem 11 in Section 4.5, rank $B = \dim \text{Col } B \leq \dim \text{Nul } A$. By this inequality and the Rank Theorem applied to A,

$$n = \text{rank } A + \dim \text{Nul } A \geq \text{rank } A + \text{rank } B$$

16. Suppose that rank $A = r_1$ and rank $B = r_2$. Then there are rank factorizations $A = C_1 R_1$ and $B = C_2 R_2$ of A and B, where C_1 is $m \times r_1$ with rank r_1, C_2 is $m \times r_2$ with rank r_2, R_1 is $r_1 \times n$ with rank r_1, and R_2 is $r_2 \times n$ with rank r_2. Create an $m \times (r_1 + r_2)$ matrix $C = [C_1 \quad C_2]$ and an $(r_1 + r_2) \times n$ matrix R by stacking R_1 over R_2. Then

$$A + B = C_1 R_1 + C_2 R_2 = [C_1 \quad C_2]\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = CR$$

Since the matrix CR is a product, its rank cannot exceed the rank of either of its factors by Exercise 12. Since C has $r_1 + r_2$ columns, the rank of C cannot exceed $r_1 + r_2$. Likewise R has $r_1 + r_2$ rows, so the rank of R cannot exceed $r_1 + r_2$. Thus the rank of $A + B$ cannot exceed $r_1 + r_2 = \text{rank } A + \text{rank } B$, or rank $(A + B) \leq$ rank A + rank B.

17. Let A be an $m \times n$ matrix with rank r.

(a) Let A_1 consist of the r pivot columns of A. The columns of A_1 are linearly independent, so A_1 is an $m \times r$ matrix with rank r.

(b) By the Rank Theorem applied to A_1, the dimension of RowA_1 is r, so A_1 has r linearly independent rows. Let A_2 consist of the r linearly independent rows of A_1. Then A_2 is an $r \times r$ matrix with linearly independent rows. By the Invertible Matrix Theorem, A_2 is invertible.

18. Let A be a 4×4 matrix and B be a 4×2 matrix, and let $\mathbf{u}_0, \ldots, \mathbf{u}_3$ be a sequence of input vectors in $\mathbb{R}^2$.

a. Use the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$ for $k = 0, \ldots, 4$, with $\mathbf{x}_0 = \mathbf{0}$.

$$\mathbf{x}_1 = A\mathbf{x}_0 + B\mathbf{u}_0 = B\mathbf{u}_0$$

$$\mathbf{x}_2 = A\mathbf{x}_1 + B\mathbf{u}_1 = AB\mathbf{u}_0 + B\mathbf{u}_1$$

$$\mathbf{x}_3 = A\mathbf{x}_2 + B\mathbf{u}_2 = A(AB\mathbf{u}_0 + B\mathbf{u}_1) + B\mathbf{u}_2 = A^2B\mathbf{u}_0 + AB\mathbf{u}_1 + B\mathbf{u}_2$$

$$\begin{aligned}\mathbf{x}_4 &= A\mathbf{x}_3 + B\mathbf{u}_3 = A(A^2B\mathbf{u}_0 + AB\mathbf{u}_1 + B\mathbf{u}_2) + B\mathbf{u}_3 \\ &= A^3B\mathbf{u}_0 + A^2B\mathbf{u}_1 + AB\mathbf{u}_2 + B\mathbf{u}_3 \\ &= \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}\begin{bmatrix}\mathbf{u}_3\\ \mathbf{u}_2\\ \mathbf{u}_1\\ \mathbf{u}_0\end{bmatrix} = M\mathbf{u}\end{aligned}$$

Note that M has 4 rows because B does, and that M has 8 columns because B and each of the matrices A^kB have 2 columns. The vector $\mathbf{u}$ in the final equation is in $\mathbb{R}^8$, because each $\mathbf{u}_k$ is in $\mathbb{R}^2$.

b. If (A, B) is controllable, then the controllability matrix has rank 4, with a pivot in each row, and the columns of M span $\mathbb{R}^4$. Therefore, for any vector $\mathbf{v}$ in $\mathbb{R}^4$, there is a vector $\mathbf{u}$ in $\mathbb{R}^8$ such that $\mathbf{v} = M\mathbf{u}$. However, from part a. we know that $\mathbf{x}_4 = M\mathbf{u}$ when $\mathbf{u}$ is partitioned into a control sequence $\mathbf{u}_0,\ldots,\mathbf{u}_3$. This particular control sequence makes $\mathbf{x}_4 = \mathbf{v}$.

19. To determine if the matrix pair (A, B) is controllable, we compute the rank of the matrix $\begin{bmatrix} B & AB & A^2B \end{bmatrix}$. To find the rank, we row reduce:

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -.9 & .81 \\ 1 & .5 & .25 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The rank of the matrix is 3, and the pair (A, B) is controllable.

20. To determine if the matrix pair (A, B) is controllable, we compute the rank of the matrix $\begin{bmatrix} B & AB & A^2B \end{bmatrix}$. To find the rank, we note that :

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & .5 & .19 \\ 1 & .7 & .45 \\ 0 & 0 & 0 \end{bmatrix}.$$

The rank of the matrix must be less than 3, and the pair (A, B) is not controllable.

21. **[M]** To determine if the matrix pair (A, B) is controllable, we compute the rank of the matrix $\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$. To find the rank, we row reduce:

$$\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1.6 \\ 0 & -1 & 1.6 & -.96 \\ -1 & 1.6 & -.96 & -.024 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1.6 \\ 0 & 0 & 1 & -1.6 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of the matrix is 3, and the pair (A, B) is not controllable.

22. **[M]** To determine if the matrix pair (A, B) is controllable, we compute the rank of the matrix $\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$. To find the rank, we row reduce:

$$\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & .5 \\ 0 & -1 & .5 & 11.45 \\ -1 & .5 & 11.45 & -10.275 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The rank of the matrix is 4, and the pair (A, B) is controllable.