38. We are given that $[\mathbf{x}]_B = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/3 \end{bmatrix}$, where $B = \left\{ \begin{bmatrix} 2.6 \\ -1.5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4.8 \end{bmatrix} \right\}$. To find the coordinates of **x** relative to the standard basis in $\mathbb{R}^3$, we must find **x**. We compute that

$$\mathbf{x} = P_B[\mathbf{x}]_B = \begin{bmatrix} 2.6 & 0 & 0 \\ -1.5 & 3 & 0 \\ 0 & 0 & 4.8 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1.3 \\ 0.75 \\ 1.6 \end{bmatrix}.$$

4.5 SOLUTIONS

Notes: Theorem 9 is true because a vector space isomorphic to $\mathbb{R}^n$ has the same algebraic properties as $\mathbb{R}^n$; a proof of this result may not be needed to convince the class. The proof of Theorem 9 relies upon the fact that the coordinate mapping is a linear transformation (which is Theorem 8 in Section 4.4). If you have skipped this result, you can prove Theorem 9 as is done in *Introduction to Linear Algebra* by Serge Lang (Springer-Verlag, New York, 1986). There are two separate groups of true-false questions in this section; the second batch is more theoretical in nature. Example 4 is useful to get students to visualize subspaces of different dimensions, and to see the relationships between subspaces of different dimensions. Exercises 31 and 32 investigate the relationship between the dimensions of the domain and the range of a linear transformation; Exercise 32 is mentioned in the proof of Theorem 17 in Section 4.8.

1. This subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and is thus a basis for *H*. Hence the dimension of *H* is 2.

2. This subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and is thus a basis for *H*. Hence the dimension of *H* is 2.

3. This subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}$. Theorem 4 in Section 4.3 can be used to show that this set is linearly independent: $\mathbf{v}_1 \neq \mathbf{0}$, $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$, and (since its first entry is not zero) $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent and is thus a basis for *H*. Alternatively, one can show that this set is linearly independent by row reducing the matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{0}]$. Hence the dimension of the subspace is 3.

4. This subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and is thus a basis for *H*. Hence the dimension of *H* is 2.

5. This subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 5 \\ 2 \\ 6 \end{bmatrix}$. The matrix *A* with these vectors as its columns row reduces to $\begin{bmatrix} 1 & -2 & 0 \\ 2 & 0 & 5 \\ 0 & -2 & 2 \\ -3 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. There is a pivot in each column, so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent and is thus a basis for *H*. Hence the dimension of *H* is 3.

6. This subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -7 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 6 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ -3 \\ 5 \\ 1 \end{bmatrix}$. The matrix *A* with these vectors as its columns row reduces to $\begin{bmatrix} 3 & 0 & -1 \\ 0 & -1 & -3 \\ -7 & 6 & 5 \\ -3 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. There is a pivot in each column, so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent and is thus a basis for *H*. Hence the dimension of *H* is 3.

7. This subspace is *H* = Nul *A*, where $A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & -1 \end{bmatrix}$. Since $[A \quad \mathbf{0}] \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, the homogeneous system has only the trivial solution. Thus $H = \text{Nul } A = \{\mathbf{0}\}$, and the dimension of *H* is 0.

8. From the equation $a - 3b + c = 0$, it is seen that $(a, b, c, d) = b(3, 1, 0, 0) + c(-1, 0, 1, 0) + d(0, 0, 0, 1)$. Thus the subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = (3,1,0,0)$, $\mathbf{v}_2 = (-1,0,1,0)$, and $\mathbf{v}_3 = (0,0,0,1)$. It is easily checked that this set of vectors is linearly independent, either by appealing to Theorem 4 in Section 4.3, or by row reducing $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{0}]$. Hence the dimension of the subspace is 3.

9. This subspace is $H = \left\{ \begin{bmatrix} a \\ b \\ a \end{bmatrix} : a,b \text{ in } \mathbb{R} \right\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and is thus a basis for H. Hence the dimension of H is 2.

10. The matrix A with these vectors as its columns row reduces to

$$\begin{bmatrix} 1 & -2 & -3 \\ -5 & 10 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

There is one pivot column, so the dimension of Col A (which is the dimension of H) is 1.

11. The matrix A with these vectors as its columns row reduces to

$$\begin{bmatrix} 1 & 3 & -2 & 5 \\ 0 & 1 & -1 & 2 \\ 2 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

There are three pivot columns, so the dimension of Col A (which is the dimension of the subspace spanned by the vectors) is 3.

12. The matrix A with these vectors as its columns row reduces to

$$\begin{bmatrix} 1 & -3 & -2 & -3 \\ -2 & -6 & 3 & 5 \\ 0 & 6 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

There are three pivot columns, so the dimension of Col A (which is the dimension of the subspace spanned by the vectors) is 3.

13. The matrix A is in echelon form. There are three pivot columns, so the dimension of Col A is 3. There are two columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has two free variables. Thus the dimension of Nul A is 2.

14. The matrix A is in echelon form. There are four pivot columns, so the dimension of Col A is 4. There are three columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has three free variables. Thus the dimension of Nul A is 3.

15. The matrix A is in echelon form. There are three pivot columns, so the dimension of Col A is 3. There are two columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has two free variables. Thus the dimension of Nul A is 2.

16. The matrix A row reduces to

$$\begin{bmatrix} 3 & 2 \\ -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

There are two pivot columns, so the dimension of Col A is 2. There are no columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$. Thus Nul $A = \{\mathbf{0}\}$, and the dimension of Nul A is 0.

17. The matrix A is in echelon form. There are three pivot columns, so the dimension of Col A is 3. There are no columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$. Thus Nul $A = \{\mathbf{0}\}$, and the dimension of Nul A is 0.

18. The matrix A is in echelon form. There are two pivot columns, so the dimension of Col A is 2. There is one column without a pivot, so the equation $A\mathbf{x} = \mathbf{0}$ has one free variable. Thus the dimension of Nul A is 1.

19. **a**. True. See the box before Example 5.
 b. False. The plane must pass through the origin; see Example 4.
 c. False. The dimension of $\mathbb{P}_n$ is $n + 1$; see Example 1.
 d. False. The set S must also have n elements; see Theorem 12.
 e. True. See Theorem 9.

20. **a**. False. The set $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$.
 b. False. The number of **free** variables is equal to the dimension of Nul A; see the box before Example 5.
 c. False. A basis could still have only finitely many elements, which would make the vector space finite-dimensional.
 d. False. The set S must also have n elements; see Theorem 12.
 e. True. See Example 4.

21. The matrix whose columns are the coordinate vectors of the Hermite polynomials relative to the standard basis $\{1, t, t^2, t^3\}$ of $\mathbb{P}_3$ is

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}.$$

This matrix has 4 pivots, so its columns are linearly independent. Since their coordinate vectors form a linearly independent set, the Hermite polynomials themselves are linearly independent in $\mathbb{P}_3$. Since there are four Hermite polynomials and dim $\mathbb{P}_3 = 4$, the Basis Theorem states that the Hermite polynomials form a basis for $\mathbb{P}_3$.

22. The matrix whose columns are the coordinate vectors of the Laguerre polynomials relative to the standard basis $\{1, t, t^2, t^3\}$ of $\mathbb{P}_3$ is

$$A = \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

This matrix has 4 pivots, so its columns are linearly independent. Since their coordinate vectors form a linearly independent set, the Laguerre polynomials themselves are linearly independent in $\mathbb{P}_3$. Since there are four Laguerre polynomials and dim $\mathbb{P}_3 = 4$, the Basis Theorem states that the Laguerre polynomials form a basis for $\mathbb{P}_3$.

23. The coordinates of $\mathbf{p}(t) = -1 + 8t^2 + 8t^3$ with respect to B satisfy

$$c_1(1) + c_2(2t) + c_3(-2 + 4t^2) + c_4(-12t + 8t^3) = -1 + 8t^2 + 8t^3$$

Equating coefficients of like powers of t produces the system of equations

$$\begin{array}{rcrcrcrcr} c_1 & & & - & 2c_3 & & & = & -1 \\ & & 2c_2 & & & - & 12c_4 & = & 0 \\ & & & & 4c_3 & & & = & 8 \\ & & & & & & 8c_4 & = & 8 \end{array}$$

Solving this system gives $c_1 = 3,\ c_2 = 6,\ c_3 = 2,\ c_4 = 1,$ and $[\mathbf{p}]_B = \begin{bmatrix} 3 \\ 6 \\ 2 \\ 1 \end{bmatrix}$.

24. The coordinates of $\mathbf{p}(t) = 5 + 5t - 2t^2$ with respect to B satisfy

$$c_1(1) + c_2(1-t) + c_3(2 - 4t + t^2) = 5 + 5t - 2t^2$$

Equating coefficients of like powers of t produces the system of equations

$$\begin{array}{rcrcrcr} c_1 & + & c_2 & + & 2c_3 & = & 5 \\ & & -c_2 & - & 4c_3 & = & 5 \\ & & & & c_3 & = & -2 \end{array}$$

Solving this system gives $c_1 = 6,\ c_2 = 3,\ c_3 = -2,$ and $[\mathbf{p}]_B = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}$.

25. Note first that $n \ge 1$ since S cannot have fewer than 1 vector. Since $n \ge 1$, $V \ne \mathbf{0}$. Suppose that S spans V and that S contains fewer than n vectors. By the Spanning Set Theorem, some subset S' of S is a basis for V. Since S contains fewer than n vectors, and S' is a subset of S, S' also contains fewer than n vectors. Thus there is a basis S' for V with fewer than n vectors, but this is impossible by Theorem 10 since $\dim V = n$. Thus S cannot span V.

26. If $\dim V = \dim H = 0$, then $V = \{\mathbf{0}\}$ and $H = \{\mathbf{0}\}$, so $H = V$. Suppose that $\dim V = \dim H > 0$. Then H contains a basis S consisting of n vectors. But applying the Basis Theorem to V, S is also a basis for V. Thus $H = V = \text{Span}S$.

27. Suppose that $\dim \mathbb{P} = k < \infty$. Now $\mathbb{P}_n$ is a subspace of $\mathbb{P}$ for all n, and $\dim \mathbb{P}_{k-1} = k$, so $\dim \mathbb{P}_{k-1} = \dim \mathbb{P}$. This would imply that $\mathbb{P}_{k-1} = \mathbb{P}$, which is clearly untrue: for example, $\mathbf{p}(t) = t^k$ is in $\mathbb{P}$ but not in $\mathbb{P}_{k-1}$. Thus the dimension of $\mathbb{P}$ cannot be finite.

28. The space $C(\mathbb{R})$ contains $\mathbb{P}$ as a subspace. If $C(\mathbb{R})$ were finite-dimensional, then $\mathbb{P}$ would also be finite-dimensional by Theorem 11. But $\mathbb{P}$ is infinite-dimensional by Exercise 27, so $C(\mathbb{R})$ must also be infinite-dimensional.

29. **a.** True. Apply the Spanning Set Theorem to the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and produce a basis for V. This basis will not have more than p elements in it, so $\dim V \le p$.

b. True. By Theorem 11, $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ can be expanded to find a basis for V. This basis will have at least p elements in it, so $\dim V \ge p$.

c. True. Take any basis (which will contain p vectors) for V and adjoin the zero vector to it.

30. **a.** False. For a counterexample, let $\mathbf{v}$ be a non-zero vector in $\mathbb{R}^3$, and consider the set $\{\mathbf{v}, 2\mathbf{v}\}$. This is a linearly dependent set in $\mathbb{R}^3$, but $\dim \mathbb{R}^3 = 3 > 2$.

b. True. If $\dim V \le p$, there is a basis for V with p or fewer vectors. This basis would be a spanning set for V with p or fewer vectors. If necessary, vectors in V could be added to this spanning set to give a spanning set for V with exactly p vectors, which contradicts the assumption.

c. False. For a counterexample, let $\mathbf{v}$ be a non-zero vector in $\mathbb{R}^3$, and consider the set $\{\mathbf{v}, 2\mathbf{v}\}$. This is a linearly dependent set in $\mathbb{R}^3$ with $3 - 1 = 2$ vectors, and $\dim \mathbb{R}^3 = 3$.

31. Since H is a nonzero subspace of a finite-dimensional vector space V, H is finite-dimensional and has a basis. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be a basis for H. We show that the set $\{T(\mathbf{u}_1), \ldots, T(\mathbf{u}_p)\}$ spans $T(H)$. Let $\mathbf{y}$ be in $T(H)$. Then there is a vector $\mathbf{x}$ in H with $T(\mathbf{x}) = \mathbf{y}$. Since $\mathbf{x}$ is in H and $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is a basis for H, $\mathbf{x}$ may be written as $\mathbf{x} = c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p$ for some scalars $c_1, \ldots, c_p$. Since the transformation T is linear,

$$\mathbf{y} = T(\mathbf{x}) = T(c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p) = c_1T(\mathbf{u}_1) + \ldots + c_pT(\mathbf{u}_p)$$

Thus $\mathbf{y}$ is a linear combination of $T(\mathbf{u}_1), \ldots, T(\mathbf{u}_p)$, and $\{T(\mathbf{u}_1), \ldots, T(\mathbf{u}_p)\}$ spans $T(H)$. By the Spanning Set Theorem, this set contains a basis for $T(H)$. This basis then has not more than p vectors, and $\dim T(H) \le p = \dim H$.

32. Since H is a nonzero subspace of a finite-dimensional vector space V, H is finite-dimensional and has a basis. Let $\{\mathbf{u}_1, \ldots \mathbf{u}_p\}$ be a basis for H. In Exercise 31 above it was shown that $\{T(\mathbf{u}_1), \ldots, T(\mathbf{u}_p)\}$ spans $T(H)$. In Exercise 32 in Section 4.3, it was shown that $\{T(\mathbf{u}_1), \ldots, T(\mathbf{u}_p)\}$ is linearly independent. Thus $\{T(\mathbf{u}_1), \ldots, T(\mathbf{u}_p)\}$ is a basis for $T(H)$, and $\dim T(H) = p = \dim H$.

33. **[M]**

a. To find a basis for $\mathbb{R}^5$ which contains the given vectors, we row reduce

$$\begin{bmatrix} -9 & 9 & 6 & 1 & 0 & 0 & 0 & 0 \\ -7 & 4 & 7 & 0 & 1 & 0 & 0 & 0 \\ 8 & 1 & -8 & 0 & 0 & 1 & 0 & 0 \\ -5 & 6 & 5 & 0 & 0 & 0 & 1 & 0 \\ 7 & -7 & -7 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/3 & 0 & 0 & 1 & 3/7 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 5/7 \\ 0 & 0 & 1 & -1/3 & 0 & 0 & 0 & -3/7 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & 22/7 \\ 0 & 0 & 0 & 0 & 0 & 1 & -9 & -53/7 \end{bmatrix}.$$

The first, second, third, fifth, and sixth columns are pivot columns, so these columns of the original matrix ($\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_2, \mathbf{e}_3\}$) form a basis for $\mathbb{R}^5$: