Math 260 Homework 4.1

4 Vector Spaces

4.1 SOLUTIONS

Notes: This section is designed to avoid the standard exercises in which a student is asked to check ten axioms on an array of sets. Theorem 1 provides the main homework tool in this section for showing that a set is a subspace. Students should be taught how to check the closure axioms. The exercises in this section (and the next few sections) emphasize $\mathbb{R}^n$, to give students time to absorb the abstract concepts. Other vectors do appear later in the chapter: the space $\mathbb{S}$ of signals is used in Section 4.8, and the spaces $\mathbb{P}_n$ of polynomials are used in many sections of Chapters 4 and 6.

1. a. If **u** and **v** are in V, then their entries are nonnegative. Since a sum of nonnegative numbers is nonnegative, the vector **u** + **v** has nonnegative entries. Thus **u** + **v** is in V.

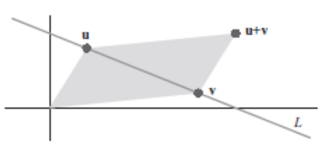
b. *Example:* If $\mathbf{u}=\begin{bmatrix}2\\2\end{bmatrix}$ and $c = -1$, then **u** is in V but $c\mathbf{u}$ is not in V.

2. a. If $\mathbf{u}=\begin{bmatrix}x\\y\end{bmatrix}$ is in W, then the vector $c\mathbf{u}=c\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}cx\\cy\end{bmatrix}$ is in W because $(cx)(cy)=c^2(xy)\ge 0$ since $xy \ge 0$.

b. *Example:* If $\mathbf{u}=\begin{bmatrix}-1\\-7\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}2\\3\end{bmatrix}$, then **u** and **v** are in W but **u** + **v** is not in W.

3. *Example:* If $\mathbf{u}=\begin{bmatrix}.5\\.5\end{bmatrix}$ and $c = 4$, then **u** is in H but $c\mathbf{u}$ is not in H. Since H is not closed under scalar multiplication, H is not a subspace of $\mathbb{R}^2$.

4. Note that **u** and **v** are on the line L, but **u** + **v** is not.



5. Yes. Since the set is Span$\{t^2\}$, the set is a subspace by Theorem 1.

6. No. The zero vector is not in the set.

7. No. The set is not closed under multiplication by scalars which are not integers.

8. Yes. The zero vector is in the set H. If **p** and **q** are in H, then $(\mathbf{p}+\mathbf{q})(0)=\mathbf{p}(0)+\mathbf{q}(0)=0+0=0$, so **p** + **q** is in H. For any scalar c, $(c\mathbf{p})(0)=c\cdot\mathbf{p}(0)=c\cdot 0=0$, so $c\mathbf{p}$ is in H. Thus H is a subspace by Theorem 1.

9. The set $H = \text{Span}\{\mathbf{v}\}$, where $\mathbf{v}=\begin{bmatrix}-2\\5\\3\end{bmatrix}$. Thus H is a subspace of $\mathbb{R}^3$ by Theorem 1.

10. The set $H = \text{Span}\{\mathbf{v}\}$, where $\mathbf{v}=\begin{bmatrix}3\\0\\-7\end{bmatrix}$. Thus H is a subspace of $\mathbb{R}^3$ by Theorem 1.

11. The set $W = \text{Span}\{\mathbf{u},\mathbf{v}\}$, where $\mathbf{u}=\begin{bmatrix}2\\-1\\0\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}3\\0\\2\end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^3$ by Theorem 1.

12. The set $W = \text{Span}\{\mathbf{u},\mathbf{v}\}$, where $\mathbf{u}=\begin{bmatrix}2\\2\\2\\0\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}4\\0\\-3\\5\end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^4$ by Theorem 1.

13. a. The vector **w** is not in the set $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$. There are 3 vectors in the set $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$.

b. The set Span$\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ contains infinitely many vectors.

c. The vector **w** is in the subspace spanned by $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ if and only if the equation $x_1\mathbf{v}_1+x_2\mathbf{v}_2+x_3\mathbf{v}_3=\mathbf{w}$ has a solution. Row reducing the augmented matrix for this system of linear equations gives

$$\begin{bmatrix}1&2&4&3\\0&1&2&1\\-1&3&6&2\end{bmatrix}\sim\begin{bmatrix}1&0&0&1\\0&1&2&1\\0&0&0&0\end{bmatrix},$$

so the equation has a solution and **w** is in the subspace spanned by $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$.

14. The augmented matrix is found as in Exercise 13c. Since

$$\begin{bmatrix}1&2&4&1\\0&1&2&3\\-1&3&6&14\end{bmatrix}\sim\begin{bmatrix}1&0&0&-5\\0&1&2&3\\0&0&0&0\end{bmatrix},$$

the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{w}$ has a solution, the vector **w** is in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

15. Since the zero vector is not in *W*, *W* is not a vector space.

16. Since the zero vector is not in *W*, *W* is not a vector space.

17. Since a vector **w** in *W* may be written as

$$\mathbf{w} = a\begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 3 \\ 0 \\ 3 \end{bmatrix} + c\begin{bmatrix} 0 \\ -1 \\ 3 \\ 0 \end{bmatrix}$$

$$S = \left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \\ 0 \end{bmatrix} \right\}$$

is a set that spans *W*.

18. Since a vector **w** in *W* may be written as

$$\mathbf{w} = a\begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 3 \\ 0 \\ 3 \\ 3 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

$$S = \left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$$

is a set that spans *W*.

19. Let *H* be the set of all functions described by $y(t) = c_1\cos\omega t + c_2\sin\omega t$. Then *H* is a subset of the vector space *V* of all real-valued functions, and may be written as $H = \text{Span}\{\cos \omega t, \sin \omega t\}$. By Theorem 1, *H* is a subspace of *V* and is hence a vector space.

20. **a.** The following facts about continuous functions must be shown.

1. The constant function $\mathbf{f}(t) = 0$ is continuous.
2. The sum of two continuous functions is continuous.
3. A constant multiple of a continuous function is continuous.

b. Let $H = \{\mathbf{f} \text{ in } C[a, b]: \mathbf{f}(a) = \mathbf{f}(b)\}$.

1. Let $\mathbf{g}(t) = 0$ for all *t* in $[a, b]$. Then $\mathbf{g}(a) = \mathbf{g}(b) = 0$, so **g** is in *H*.
2. Let **g** and **h** be in *H*. Then $\mathbf{g}(a) = \mathbf{g}(b)$ and $\mathbf{h}(a) = \mathbf{h}(b)$, and $(\mathbf{g} + \mathbf{h})(a) = \mathbf{g}(a) + \mathbf{h}(a) = \mathbf{g}(b) + \mathbf{h}(b) = (\mathbf{g} + \mathbf{h})(b)$, so $\mathbf{g} + \mathbf{h}$ is in *H*.
3. Let **g** be in *H*. Then $\mathbf{g}(a) = \mathbf{g}(b)$, and $(c\mathbf{g})(a) = c\mathbf{g}(a) = c\mathbf{g}(b) = (c\mathbf{g})(b)$, so $c\mathbf{g}$ is in *H*.

Thus *H* is a subspace of $C[a, b]$.

21. The set *H* is a subspace of $M_{2\times 2}$. The zero matrix is in *H*, the sum of two upper triangular matrices is upper triangular, and a scalar multiple of an upper triangular matrix is upper triangular.

22. The set *H* is a subspace of $M_{2\times 4}$. The 2×4 zero matrix 0 is in *H* because $F0 = 0$. If *A* and *B* are matrices in *H*, then $F(A + B) = FA + FB = 0 + 0 = 0$, so $A + B$ is in *H*. If *A* is in *H* and *c* is a scalar, then $F(cA) = c(FA) = c0 = 0$, so *cA* is in *H*.

23. **a.** False. The zero vector in *V* is the function **f** whose values $\mathbf{f}(t)$ are zero *for all t* in $\mathbb{R}$.
b. False. An arrow in three-dimensional space is an example of a vector, but not every arrow is a vector.
c. False. See Exercises 1, 2, and 3 for examples of subsets which contain the zero vector but are not subspaces.
d. True. See the paragraph before Example 6.
e. False. Digital signals are used. See Example 3.

24. **a.** True. See the definition of a vector space.
b. True. See statement (3) in the box before Example 1.
c. True. See the paragraph before Example 6.
d. False. See Example 8.
e. False. The second and third parts of the conditions are stated incorrectly. For example, part (ii) does not state that **u** and **v** represent all possible elements of *H*.

25. 2, 4

26. **a.** 3
b. 5
c. 4

27. **a.** 8
b. 3
c. 5
d. 4

28. **a.** 4
b. 7
c. 3
d. 5
e. 4

29. Consider $\mathbf{u} + (-1)\mathbf{u}$. By Axiom 10, $\mathbf{u} + (-1)\mathbf{u} = 1\mathbf{u} + (-1)\mathbf{u}$. By Axiom 8, $1\mathbf{u} + (-1)\mathbf{u} = (1 + (-1))\mathbf{u} = 0\mathbf{u}$. By Exercise 27, $0\mathbf{u} = \mathbf{0}$. Thus $\mathbf{u} + (-1)\mathbf{u} = \mathbf{0}$, and by Exercise 26 $(-1)\mathbf{u} = -\mathbf{u}$.

30. By Axiom 10 $\mathbf{u} = 1\mathbf{u}$. Since *c* is nonzero, $c^{-1}c = 1$, and $\mathbf{u} = (c^{-1}c)\mathbf{u}$. By Axiom 9, $(c^{-1}c)\mathbf{u} = c^{-1}(c\mathbf{u}) = c^{-1}\mathbf{0}$ since $c\mathbf{u} = \mathbf{0}$. Thus $\mathbf{u} = c^{-1}\mathbf{0} = \mathbf{0}$ by Property (2), proven in Exercise 28.

31. Any subspace H that contains **u** and **v** must also contain all scalar multiples of **u** and **v**, and hence must also contain all sums of scalar multiples of **u** and **v**. Thus H must contain all linear combinations of **u** and **v**, or Span$\{\mathbf{u}, \mathbf{v}\}$.

Note: Exercises 32–34 provide good practice for mathematics majors because these arguments involve simple symbol manipulation typical of mathematical proofs. Most students outside mathematics might profit more from other types of exercises.

32. Both H and K contain the zero vector of V because they are subspaces of V. Thus the zero vector of V is in $H \cap K$. Let **u** and **v** be in $H \cap K$. Then **u** and **v** are in H. Since H is a subspace $\mathbf{u} + \mathbf{v}$ is in H. Likewise **u** and **v** are in K. Since K is a subspace $\mathbf{u} + \mathbf{v}$ is in K. Thus $\mathbf{u} + \mathbf{v}$ is in $H \cap K$. Let **u** be in $H \cap K$. Then **u** is in H. Since H is a subspace $c\mathbf{u}$ is in H. Likewise **u** is in K. Since K is a subspace $c\mathbf{u}$ is in K. Thus $c\mathbf{u}$ is in $H \cap K$ for any scalar c, and $H \cap K$ is a subspace of V.

The union of two subspaces is not in general a subspace. For an example in $\mathbb{R}^2$ let H be the x-axis and let K be the y-axis. Then both H and K are subspaces of $\mathbb{R}^2$, but $H \cup K$ is not closed under vector addition. The subset $H \cup K$ is thus not a subspace of $\mathbb{R}^2$.

33. a. Given subspaces H and K of a vector space V, the zero vector of V belongs to $H + K$, because **0** is in both H and K (since they are subspaces) and $\mathbf{0} = \mathbf{0} + \mathbf{0}$. Next, take two vectors in $H + K$, say $\mathbf{w}_1 = \mathbf{u}_1 + \mathbf{v}_1$ and $\mathbf{w}_2 = \mathbf{u}_2 + \mathbf{v}_2$ where $\mathbf{u}_1$ and $\mathbf{u}_2$ are in H, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are in K. Then

$$\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{u}_1 + \mathbf{v}_1 + \mathbf{u}_2 + \mathbf{v}_2 = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2)$$

because vector addition in V is commutative and associative. Now $\mathbf{u}_1 + \mathbf{u}_2$ is in H and $\mathbf{v}_1 + \mathbf{v}_2$ is in K because H and K are subspaces. This shows that $\mathbf{w}_1 + \mathbf{w}_2$ is in $H + K$. Thus $H + K$ is closed under addition of vectors. Finally, for any scalar c,

$$c\mathbf{w}_1 = c(\mathbf{u}_1 + \mathbf{v}_1) = c\mathbf{u}_1 + c\mathbf{v}_1$$

The vector $c\mathbf{u}_1$ belongs to H and $c\mathbf{v}_1$ belongs to K, because H and K are subspaces. Thus, $c\mathbf{w}_1$ belongs to $H + K$, so $H + K$ is closed under multiplication by scalars. These arguments show that $H + K$ satisfies all three conditions necessary to be a subspace of V.

b. Certainly H is a subset of $H + K$ because every vector **u** in H may be written as $\mathbf{u} + \mathbf{0}$, where the zero vector **0** is in K (and also in H, of course). Since H contains the zero vector of $H + K$, and H is closed under vector addition and multiplication by scalars (because H is a subspace of V), H is a subspace of $H + K$. The same argument applies when H is replaced by K, so K is also a subspace of $H + K$.

34. A proof that $H + K = \text{Span}\{\mathbf{u}_1,\ldots,\mathbf{u}_p,\mathbf{v}_1,\ldots,\mathbf{v}_q\}$ has two parts. First, one must show that $H + K$ is a subset of $\text{Span}\{\mathbf{u}_1,\ldots,\mathbf{u}_p,\mathbf{v}_1,\ldots,\mathbf{v}_q\}$. Second, one must show that $\text{Span}\{\mathbf{u}_1,\ldots,\mathbf{u}_p,\mathbf{v}_1,\ldots,\mathbf{v}_q\}$ is a subset of $H + K$.

(1) A typical vector in H has the form $c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p$ and a typical vector in K has the form $d_1\mathbf{v}_1 + \ldots + d_q\mathbf{v}_q$. The sum of these two vectors is a linear combination of $\mathbf{u}_1,\ldots,\mathbf{u}_p,\mathbf{v}_1,\ldots,\mathbf{v}_q$ and so belongs to $\text{Span}\{\mathbf{u}_1,\ldots,\mathbf{u}_p,\mathbf{v}_1,\ldots,\mathbf{v}_q\}$. Thus $H + K$ is a subset of $\text{Span}\{\mathbf{u}_1,\ldots,\mathbf{u}_p,\mathbf{v}_1,\ldots,\mathbf{v}_q\}$.

(2) Each of the vectors $\mathbf{u}_1,\ldots,\mathbf{u}_p,\mathbf{v}_1,\ldots,\mathbf{v}_q$ belongs to $H + K$, by Exercise 33(b), and so any linear combination of these vectors belongs to $H + K$, since $H + K$ is a subspace, by Exercise 33(a). Thus, $\text{Span}\{\mathbf{u}_1,\ldots,\mathbf{u}_p,\mathbf{v}_1,\ldots,\mathbf{v}_q\}$ is a subset of $H + K$.

35. [M] Since

$$\begin{bmatrix} 8 & -4 & -7 & 9 \\ -4 & 3 & 6 & -4 \\ -3 & -2 & -5 & -4 \\ 9 & -8 & -18 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

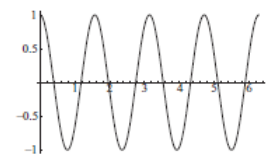
w is in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

36. [M] Since

$$[A \quad \mathbf{y}] = \begin{bmatrix} 3 & -5 & -9 & -4 \\ 8 & 7 & -6 & -8 \\ -5 & -8 & 3 & 6 \\ 2 & -2 & -9 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/5 \\ 0 & 1 & 0 & -2/5 \\ 0 & 0 & 1 & 3/5 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

y is in the subspace spanned by the columns of A.

37. [M] The graph of $\mathbf{f}(t)$ is given below. A conjecture is that $\mathbf{f}(t) = \cos 4t$.



The graph of $\mathbf{g}(t)$ is given below. A conjecture is that $\mathbf{g}(t) = \cos 6t$.