Math 260 Homework 3.3

3.3 SOLUTIONS

Notes: This section features several independent topics from which to choose. The geometric interpretation of the determinant (Theorem 10) provides the key to changes of variables in multiple integrals. Students of economics and engineering are likely to need Cramer's Rule in later courses. Exercises 1–10 concern Cramer's Rule, exercises 11–18 deal with the adjugate, and exercises 19–32 cover the geometric interpretation of the determinant. In particular, Exercise 25 examines students' understanding of linear independence and requires a careful explanation, which is discussed in the *Study Guide*. The *Study Guide* also contains a heuristic proof of Theorem 9 for 2×2 matrices.

1. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 5 & 7 \\ 2 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. We compute

$A_1(\mathbf{b}) = \begin{bmatrix} 3 & 7 \\ 1 & 4 \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$, $\det A = 6$, $\det A_1(\mathbf{b}) = 5$, $\det A_2(\mathbf{b}) = -1$,

$x_1 = \dfrac{\det A_1(\mathbf{b})}{\det A} = \dfrac{5}{6}, x_2 = \dfrac{\det A_2(\mathbf{b})}{\det A} = -\dfrac{1}{6}.$

2. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$. We compute

$A_1(\mathbf{b}) = \begin{bmatrix} 6 & 1 \\ 7 & 2 \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix}$, $\det A = 3$, $\det A_1(\mathbf{b}) = 5$, $\det A_2(\mathbf{b}) = -2$,

$x_1 = \dfrac{\det A_1(\mathbf{b})}{\det A} = \dfrac{5}{3}, x_2 = \dfrac{\det A_2(\mathbf{b})}{\det A} = -\dfrac{2}{3}.$

3. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$. We compute

$A_1(\mathbf{b}) = \begin{bmatrix} 7 & -2 \\ -5 & 6 \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} 3 & 7 \\ -5 & -5 \end{bmatrix}$, $\det A = 8$, $\det A_1(\mathbf{b}) = 32$, $\det A_2(\mathbf{b}) = 20$,

$x_1 = \dfrac{\det A_1(\mathbf{b})}{\det A} = \dfrac{32}{8} = 4, x_2 = \dfrac{\det A_2(\mathbf{b})}{\det A} = \dfrac{20}{8} = \dfrac{5}{2}.$

4. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} -5 & 3 \\ 3 & -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 9 \\ -5 \end{bmatrix}$. We compute

$A_1(\mathbf{b}) = \begin{bmatrix} 9 & 3 \\ -5 & -1 \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} -5 & 9 \\ 3 & -5 \end{bmatrix}$, $\det A = -4$, $\det A_1(\mathbf{b}) = 6$, $\det A_2(\mathbf{b}) = -2$,

$x_1 = \dfrac{\det A_1(\mathbf{b})}{\det A} = \dfrac{6}{-4} = -\dfrac{3}{2}, x_2 = \dfrac{\det A_2(\mathbf{b})}{\det A} = \dfrac{-2}{-4} = \dfrac{1}{2}.$

5. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ -8 \\ -3 \end{bmatrix}$. We compute

$A_1(\mathbf{b}) = \begin{bmatrix} 7 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} 2 & 7 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{bmatrix}, A_3(\mathbf{b}) = \begin{bmatrix} 2 & 1 & 7 \\ -3 & 0 & -8 \\ 0 & 1 & -3 \end{bmatrix},$

$\det A = 4, \det A_1(\mathbf{b}) = 6, \det A_2(\mathbf{b}) = 16, \det A_3(\mathbf{b}) = -14,$

$x_1 = \dfrac{\det A_1(\mathbf{b})}{\det A} = \dfrac{6}{4} = \dfrac{3}{2}, x_2 = \dfrac{\det A_2(\mathbf{b})}{\det A} = \dfrac{16}{4} = 4, x_3 = \dfrac{\det A_3(\mathbf{b})}{\det A} = \dfrac{-14}{4} = -\dfrac{7}{2}.$

6. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$. We compute

$A_1(\mathbf{b}) = \begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & 1 & 3 \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} 2 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & -2 & 3 \end{bmatrix}, A_3(\mathbf{b}) = \begin{bmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 3 & 1 & -2 \end{bmatrix},$

$\det A = 4$, $\det A_1(\mathbf{b}) = -16$, $\det A_2(\mathbf{b}) = 52$, $\det A_3(\mathbf{b}) = -4$,

$x_1 = \dfrac{\det A_1(\mathbf{b})}{\det A} = \dfrac{-16}{4} = -4, x_2 = \dfrac{\det A_2(\mathbf{b})}{\det A} = \dfrac{52}{4} = 13, x_3 = \dfrac{\det A_3(\mathbf{b})}{\det A} = \dfrac{-4}{4} = -1.$

7. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 6s & 4 \\ 9 & 2s \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$. We compute

$A_1(\mathbf{b}) = \begin{bmatrix} 5 & 4 \\ -2 & 2s \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} 6s & 5 \\ 9 & -2 \end{bmatrix}$, $\det A_1(\mathbf{b}) = 10s + 8$, $\det A_2(\mathbf{b}) = -12s - 45$.

Since $\det A = 12s^2 - 36 = 12(s^2 - 3) \neq 0$ for $s \neq \pm\sqrt{3}$, the system will have a unique solution when $s \neq \pm\sqrt{3}$. For such a system, the solution will be

$x_1 = \dfrac{\det A_1(\mathbf{b})}{\det A} = \dfrac{10s + 8}{12(s^2 - 3)} = \dfrac{5s + 4}{6(s^2 - 3)}, x_2 = \dfrac{\det A_2(\mathbf{b})}{\det A} = \dfrac{-12s - 45}{12(s^2 - 3)} = \dfrac{-4s - 15}{4(s^2 - 3)}.$

8. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 3s & -5 \\ 9 & 5s \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. We compute

$A_1(\mathbf{b}) = \begin{bmatrix} 3 & -5 \\ 2 & 5s \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} 3s & 3 \\ 9 & 2 \end{bmatrix}$, $\det A_1(\mathbf{b}) = 15s + 10$, $\det A_2(\mathbf{b}) = 6s - 27$.

Since $\det A = 15s^2 + 45 = 15(s^2 + 3) \neq 0$ for all values of s, the system will have a unique solution for all values of s. For such a system, the solution will be

$x_1 = \dfrac{\det A_1(\mathbf{b})}{\det A} = \dfrac{15s + 10}{15(s^2 + 3)} = \dfrac{3s + 2}{3(s^2 + 3)}, x_2 = \dfrac{\det A_2(\mathbf{b})}{\det A} = \dfrac{6s - 27}{15(s^2 + 3)} = \dfrac{2s - 9}{5(s^2 + 3)}.$

9. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} s & -2s \\ 3 & 6s \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$. We compute

$$A_1(\mathbf{b}) = \begin{bmatrix} -1 & -2s \\ 4 & 6s \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} s & -1 \\ 3 & 4 \end{bmatrix}, \det A_1(\mathbf{b}) = 2s, \det A_2(\mathbf{b}) = 4s+3.$$

Since $\det A = 6s^2 + 6s = 6s(s+1) = 0$ for $s = 0, -1$, the system will have a unique solution when $s \neq 0$, -1. For such a system, the solution will be

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{2s}{6s(s+1)} = \frac{1}{3(s+1)}, x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{4s+3}{6s(s+1)}.$$

10. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 2s & 1 \\ 3s & 6s \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. We compute

$$A_1(\mathbf{b}) = \begin{bmatrix} 1 & 1 \\ 2 & 6s \end{bmatrix}, A_2(\mathbf{b}) = \begin{bmatrix} 2s & 1 \\ 3s & 2 \end{bmatrix}, \det A_1(\mathbf{b}) = 6s - 2, \det A_2(\mathbf{b}) = s.$$

Since $\det A = 12s^2 - 3s = 3s(4s-1) = 0$ for $s = 0, 1/4$, the system will have a unique solution when $s \neq 0, 1/4$. For such a system, the solution will be

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{6s-2}{3s(4s-1)}, x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{s}{3s(4s-1)} = \frac{1}{3(4s-1)}.$$

11. Since $\det A = 3$ and the cofactors of the given matrix are

$$C_{11} = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0, \quad C_{12} = -\begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} = -3, \quad C_{13} = \begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} = 3,$$

$$C_{21} = -\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} = 1, \quad C_{22} = \begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} = -1, \quad C_{23} = -\begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix} = 2,$$

$$C_{31} = \begin{vmatrix} -2 & -1 \\ 0 & 0 \end{vmatrix} = 0, \quad C_{32} = -\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} = -3, \quad C_{33} = \begin{vmatrix} 0 & -2 \\ 3 & 0 \end{vmatrix} = 6,$$

$$\operatorname{adj} A = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & -3 \\ 3 & 2 & 6 \end{bmatrix} \text{ and } A^{-1} = \frac{1}{\det A}\operatorname{adj} A = \begin{bmatrix} 0 & 1/3 & 0 \\ -1 & -1/3 & -1 \\ 1 & 2/3 & 2 \end{bmatrix}.$$

12. Since $\det A = 5$ and the cofactors of the given matrix are

$$C_{11} = \begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad C_{12} = -\begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} = 0, \quad C_{13} = \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} = 2,$$

$$C_{21} = -\begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} = 3, \quad C_{22} = \begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} = 0, \quad C_{23} = -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1,$$

$$C_{31} = \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = 7, \quad C_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 5, \quad C_{33} = \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} = -4,$$

$$\operatorname{adj} A = \begin{bmatrix} -1 & 3 & 7 \\ 0 & 0 & 5 \\ 2 & -1 & -4 \end{bmatrix} \text{ and } A^{-1} = \frac{1}{\det A}\operatorname{adj} A = \begin{bmatrix} -1/5 & 3/5 & 7/5 \\ 0 & 0 & 1 \\ 2/5 & -1/5 & -4/5 \end{bmatrix}.$$

13. Since $\det A = 6$ and the cofactors of the given matrix are

$$C_{11} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1, \quad C_{12} = -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1, \quad C_{13} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1,$$

$$C_{21} = -\begin{vmatrix} 5 & 4 \\ 1 & 1 \end{vmatrix} = -1, \quad C_{22} = \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = -5, \quad C_{23} = -\begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} = 7,$$

$$C_{31} = \begin{vmatrix} 5 & 4 \\ 0 & 1 \end{vmatrix} = 5, \quad C_{32} = -\begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} = 1, \quad C_{33} = \begin{vmatrix} 3 & 5 \\ 1 & 0 \end{vmatrix} = -5,$$

$$\operatorname{adj} A = \begin{bmatrix} -1 & -1 & 5 \\ 1 & -5 & 1 \\ 1 & 7 & -5 \end{bmatrix} \text{ and } A^{-1} = \frac{1}{\det A}\operatorname{adj} A = \begin{bmatrix} -1/6 & -1/6 & 5/6 \\ 1/6 & -5/6 & 1/6 \\ 1/6 & 7/6 & -5/6 \end{bmatrix}.$$

14. Since $\det A = -1$ and the cofactors of the given matrix are

$$C_{11} = \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 5, \quad C_{12} = -\begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix} = 2, \quad C_{13} = \begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} = -4,$$

$$C_{21} = -\begin{vmatrix} 6 & 7 \\ 3 & 4 \end{vmatrix} = -3, \quad C_{22} = \begin{vmatrix} 3 & 7 \\ 2 & 4 \end{vmatrix} = -2, \quad C_{23} = -\begin{vmatrix} 3 & 6 \\ 2 & 3 \end{vmatrix} = 3,$$

$$C_{31} = \begin{vmatrix} 6 & 7 \\ 2 & 1 \end{vmatrix} = -8, \quad C_{32} = -\begin{vmatrix} 3 & 7 \\ 0 & 1 \end{vmatrix} = -3, \quad C_{33} = \begin{vmatrix} 3 & 6 \\ 0 & 2 \end{vmatrix} = 6,$$

$$\operatorname{adj} A = \begin{bmatrix} 5 & -3 & -8 \\ 2 & -2 & -3 \\ -4 & 3 & 6 \end{bmatrix} \text{ and } A^{-1} = \frac{1}{\det A}\operatorname{adj} A = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}.$$

15. Since $\det A = 6$ and the cofactors of the given matrix are

$$C_{11} = \begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} = 2, \quad C_{12} = -\begin{vmatrix} -1 & 0 \\ -2 & 2 \end{vmatrix} = 2, \quad C_{13} = \begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix} = -1,$$

$$C_{21} = -\begin{vmatrix} 0 & 0 \\ 3 & 2 \end{vmatrix} = 0, \quad C_{22} = \begin{vmatrix} 3 & 0 \\ -2 & 2 \end{vmatrix} = 6, \quad C_{23} = -\begin{vmatrix} 3 & 0 \\ -2 & 3 \end{vmatrix} = -9,$$

$$C_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0, \quad C_{32} = -\begin{vmatrix} 3 & 0 \\ -1 & 0 \end{vmatrix} = 0, \quad C_{33} = \begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} = 3,$$

$$\operatorname{adj} A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ -1 & -9 & 3 \end{bmatrix} \text{ and } A^{-1} = \frac{1}{\det A}\operatorname{adj} A = \begin{bmatrix} 1/3 & 0 & 0 \\ 1/3 & 1 & 0 \\ -1/6 & -3/2 & 1/2 \end{bmatrix}.$$

16. Since $\det A = -9$ and the cofactors of the given matrix are

$$C_{11} = \begin{vmatrix} -3 & 1 \\ 0 & 3 \end{vmatrix} = -9, \quad C_{12} = -\begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} = 0, \quad C_{13} = \begin{vmatrix} 0 & -3 \\ 0 & 0 \end{vmatrix} = 0,$$

$$C_{21} = -\begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} = -6, \quad C_{22} = \begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix} = 3, \quad C_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0,$$

$$C_{31}=\begin{vmatrix}2&4\\-3&1\end{vmatrix}=14,\quad C_{32}=-\begin{vmatrix}1&4\\0&1\end{vmatrix}=-1,\quad C_{33}=\begin{vmatrix}1&2\\0&-3\end{vmatrix}=-3,$$

$$\text{adj}\,A=\begin{bmatrix}-9&-6&14\\0&3&-1\\0&0&-3\end{bmatrix}\text{ and } A^{-1}=\frac{1}{\det A}\text{adj}\,A=\begin{bmatrix}1&2/3&-14/9\\0&-1/3&1/9\\0&0&1/3\end{bmatrix}.$$

17. Let $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$. Then the cofactors of A are $C_{11}=|d|=d$, $C_{12}=-|c|=-c$, $C_{21}=-|b|=-b$, and $C_{22}=|a|=a$. Thus $\text{adj}\,A=\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$. Since $\det A=ad-bc$, Theorem 8 gives that $A^{-1}=\frac{1}{\det A}\text{adj}\,A=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$. This result is identical to that of Theorem 4 in Section 2.2.

18. Each cofactor of A is an integer since it is a sum of products of entries in A. Hence all entries in adj A will be integers. Since det A = 1, the inverse formula in Theorem 8 shows that all the entries in A^{-1} will be integers.

19. The parallelogram is determined by the columns of $A=\begin{bmatrix}5&6\\2&4\end{bmatrix}$, so the area of the parallelogram is $|\det A|=|8|=8$.

20. The parallelogram is determined by the columns of $A=\begin{bmatrix}-1&4\\3&-5\end{bmatrix}$, so the area of the parallelogram is $|\det A|=|-7|=7$.

21. First translate one vertex to the origin. For example, subtract (–1, 0) from each vertex to get a new parallelogram with vertices (0, 0),(1, 5),(2, –4), and (3, 1). This parallelogram has the same area as the original, and is determined by the columns of $A=\begin{bmatrix}1&2\\5&-4\end{bmatrix}$, so the area of the parallelogram is $|\det A|=|-14|=14$.

22. First translate one vertex to the origin. For example, subtract (0, –2) from each vertex to get a new parallelogram with vertices (0, 0),(6, 1),(–3, 3), and (3, 4). This parallelogram has the same area as the original, and is determined by the columns of $A=\begin{bmatrix}6&-3\\1&3\end{bmatrix}$, so the area of the parallelogram is $|\det A|=|21|=21$.

23. The parallelepiped is determined by the columns of $A=\begin{bmatrix}1&1&7\\0&2&1\\-2&4&0\end{bmatrix}$, so the volume of the parallelepiped is $|\det A|=|22|=22$.

24. The parallelepiped is determined by the columns of $A=\begin{bmatrix}1&-2&-1\\4&-5&2\\0&2&-1\end{bmatrix}$, so the volume of the parallelepiped is $|\det A|=|-15|=15$.

25. The Invertible Matrix Theorem says that a 3 × 3 matrix A is not invertible if and only if its columns are linearly dependent. This will happen if and only if one of the columns is a linear combination of the others; that is, if one of the vectors is in the plane spanned by the other two vectors. This is equivalent to the condition that the parallelepiped determined by the three vectors has zero volume, which is in turn equivalent to the condition that det A = 0.

26. By definition, $\mathbf{p}+S$ is the set of all vectors of the form $\mathbf{p}+\mathbf{v}$, where $\mathbf{v}$ is in S. Applying T to a typical vector in $\mathbf{p}+S$, we have $T(\mathbf{p}+\mathbf{v})=T(\mathbf{p})+T(\mathbf{v})$. This vector is in the set denoted by $T(\mathbf{p})+T(S)$. This proves that T maps the set $\mathbf{p}+S$ into the set $T(\mathbf{p})+T(S)$. Conversely, any vector in $T(\mathbf{p})+T(S)$ has the form $T(\mathbf{p})+T(\mathbf{v})$ for some $\mathbf{v}$ in S. This vector may be written as $T(\mathbf{p}+\mathbf{v})$. This shows that every vector in $T(\mathbf{p})+T(S)$ is the image under T of some point $\mathbf{p}+\mathbf{v}$ in $\mathbf{p}+S$.

27. Since the parallelogram S is determined by the columns of $\begin{bmatrix}-2&-2\\3&5\end{bmatrix}$, the area of S is $\left|\det\begin{bmatrix}-2&-2\\3&5\end{bmatrix}\right|=|-4|=4$. The matrix A has $\det A=\begin{vmatrix}6&-2\\-3&2\end{vmatrix}=6$. By Theorem 10, the area of $T(S)$ is $|\det A|\{\text{area of } S\}=6\cdot 4=24$. Alternatively, one may compute the vectors that determine the image, namely, the columns of

$$A[\mathbf{b}_1\quad \mathbf{b}_2]=\begin{bmatrix}6&-2\\-3&2\end{bmatrix}\begin{bmatrix}-2&-2\\3&5\end{bmatrix}=\begin{bmatrix}-18&-22\\12&16\end{bmatrix}$$

The determinant of this matrix is –24, so the area of the image is 24.

28. Since the parallelogram S is determined by the columns of $\begin{bmatrix}4&0\\-7&1\end{bmatrix}$, the area of S is $\left|\det\begin{bmatrix}4&0\\-7&1\end{bmatrix}\right|=|4|=4$. The matrix A has $\det A=\begin{vmatrix}7&2\\1&1\end{vmatrix}=5$. By Theorem 10, the area of $T(S)$ is $|\det A|\{\text{area of } S\}=5\cdot 4=20$. Alternatively, one may compute the vectors that determine the image, namely, the columns of

$$A[\mathbf{b}_1\quad \mathbf{b}_2]=\begin{bmatrix}7&2\\1&1\end{bmatrix}\begin{bmatrix}4&0\\-7&1\end{bmatrix}=\begin{bmatrix}14&2\\-3&1\end{bmatrix}$$

The determinant of this matrix is 20, so the area of the image is 20.

29. The area of the triangle will be one half of the area of the parallelogram determined by $\mathbf{v}_1$ and $\mathbf{v}_2$. By Theorem 9, the area of the triangle will be (1/2)|det A|, where $A=[\mathbf{v}_1\quad \mathbf{v}_2]$.

30. Translate R to a new triangle of equal area by subtracting (x_3, y_3) from each vertex. The new triangle has vertices (0, 0), (x_1-x_3, y_1-y_3), and (x_2-x_3, y_2-y_3). By Exercise 29, the area of the triangle will be

$$\frac{1}{2}\left|\det\begin{bmatrix} x_1-x_3 & x_2-x_3 \\ y_1-y_3 & y_2-y_3 \end{bmatrix}\right|.$$

Now consider using row operations and a cofactor expansion to compute the determinant in the formula:

$$\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = \det\begin{bmatrix} x_1-x_3 & y_1-y_3 & 0 \\ x_2-x_3 & y_2-y_3 & 0 \\ x_3 & y_3 & 1 \end{bmatrix} = \det\begin{bmatrix} x_1-x_3 & y_1-y_3 \\ x_2-x_3 & y_2-y_3 \end{bmatrix}$$

By Theorem 5,

$$\det\begin{bmatrix} x_1-x_3 & y_1-y_3 \\ x_2-x_3 & y_2-y_3 \end{bmatrix} = \det\begin{bmatrix} x_1-x_3 & x_2-x_3 \\ y_1-y_3 & y_2-y_3 \end{bmatrix}$$

So the above observation allows us to state that the area of the triangle will be

$$\frac{1}{2}\left|\det\begin{bmatrix} x_1-x_3 & x_2-x_3 \\ y_1-y_3 & y_2-y_3 \end{bmatrix}\right| = \frac{1}{2}\left|\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}\right|$$

31. a. To show that $T(S)$ is bounded by the ellipsoid with equation $\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}=1$, let $\mathbf{u}=\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and let $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=A\mathbf{u}$. Then $u_1=x_1/a$, $u_2=x_2/b$, and $u_3=x_3/c$, and **u** lies inside S (or $u_1^2+u_2^2+u_3^2\le 1$) if and only if **x** lies inside $T(S)$ (or $\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}\le 1$).

b. By the generalization of Theorem 10,

$$\{\text{volume of ellipsoid}\}=\{\text{volume of } T(S)\}$$
$$=|\det A|\cdot\{\text{volume of } S\}=abc\frac{4\pi}{3}=\frac{4\pi abc}{3}$$

32. a. A linear transformation T that maps S onto S' will map $\mathbf{e}_1$ to $\mathbf{v}_1$, $\mathbf{e}_2$ to $\mathbf{v}_2$, and $\mathbf{e}_3$ to $\mathbf{v}_3$; that is, $T(\mathbf{e}_1)=\mathbf{v}_1$, $T(\mathbf{e}_2)=\mathbf{v}_2$, and $T(\mathbf{e}_3)=\mathbf{v}_3$. The standard matrix for this transformation will be $A=[T(\mathbf{e}_1)\quad T(\mathbf{e}_2)\quad T(\mathbf{e}_3)]=[\mathbf{v}_1\quad \mathbf{v}_2\quad \mathbf{v}_3]$.

b. The area of the base of S is (1/2)(1)(1) = 1/2, so the volume of S is (1/3)(1/2)(1) = 1/6. By part a. $T(S) = S'$, so the generalization of Theorem 10 gives that the volume of S' is $|\det A|$\{volume of S\} = (1/6)$|\det A|$.

33. **[M]** Answers will vary. In MATLAB, entries in B – inv(A) are approximately 10^{-15} or smaller.

34. **[M]** Answers will vary, as will the commands which produce the second entry of **x**. For example, the MATLAB command is `x2 = det([A(:,1) b A(:,3:4)])/det(A)` while the Mathematica command is `x2 = Det[{Transpose[A][[1]],b,Transpose[A][[3]], Transpose[A][[4]]}]/Det[A]`.

35. **[M]** MATLAB Student Version 4.0 uses 57,771 flops for inv A and 14,269,045 flops for the inverse formula. The `inv(A)` command requires only about 0.4% of the operations for the inverse formula.

Chapter 3 SUPPLEMENTARY EXERCISES

1. a. True. The columns of A are linearly dependent.

b. True. See Exercise 30 in Section 3.2.

c. False. See Theorem 3(c); in this case $\det 5A = 5^3 \det A$.

d. False. Consider $A=\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, and $A+B=\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$.

e. False. By Theorem 6, $\det A^3 = 2^3$.

f. False. See Theorem 3(b).

g. True. See Theorem 3(c).

h. True. See Theorem 3(a).

i. False. See Theorem 5.

j. False. See Theorem 3(c); this statement is false for $n\times n$ invertible matrices with n an even integer.

k. True. See Theorems 6 and 5; $\det A^TA = (\det A)^2$.

l. False. The coefficient matrix must be invertible.

m. False. The area of the **triangle** is 5.

n. True. See Theorem 6; $\det A^3 = (\det A)^3$.

o. False. See Exercise 31 in Section 3.2.

p. True. See Theorem 6.

2. $\begin{vmatrix} 12 & 13 & 14 \\ 15 & 16 & 17 \\ 18 & 19 & 20 \end{vmatrix} = \begin{vmatrix} 12 & 13 & 14 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} = 0$

3. $\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & b+c \\ 0 & b-a & a-b \\ 0 & c-a & a-c \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & a & b+c \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{vmatrix} = 0$

4. $\begin{vmatrix} a & b & c \\ a+x & b+x & c+x \\ a+y & b+y & c+y \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & x & x \\ y & y & y \end{vmatrix} = xy\begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$

5. $\begin{vmatrix} 9 & 1 & 9 & 9 & 9 \\ 9 & 0 & 9 & 9 & 2 \\ 4 & 0 & 0 & 5 & 0 \\ 9 & 0 & 3 & 9 & 0 \\ 6 & 0 & 0 & 7 & 0 \end{vmatrix} = (-1)\begin{vmatrix} 9 & 9 & 9 & 2 \\ 4 & 0 & 5 & 0 \\ 9 & 3 & 9 & 0 \\ 6 & 0 & 7 & 0 \end{vmatrix} = (-1)(-2)\begin{vmatrix} 4 & 0 & 5 \\ 9 & 3 & 9 \\ 6 & 0 & 7 \end{vmatrix}$

$= (-1)(-2)(3)\begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} = (-1)(-2)(3)(-2) = -12$

6. $\begin{vmatrix} 4 & 8 & 8 & 8 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 6 & 8 & 8 & 8 & 7 \\ 0 & 8 & 8 & 3 & 0 \\ 0 & 8 & 2 & 0 & 0 \end{vmatrix} = (1)\begin{vmatrix} 4 & 8 & 8 & 5 \\ 6 & 8 & 8 & 7 \\ 0 & 8 & 3 & 0 \\ 0 & 2 & 0 & 0 \end{vmatrix} = (1)(2)\begin{vmatrix} 4 & 8 & 5 \\ 6 & 8 & 7 \\ 0 & 3 & 0 \end{vmatrix} = (1)(2)(-3)\begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} = (1)(2)(-3)(-2) = 12$

7. Expand along the first row to obtain

$$\begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix} = 1\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} - x\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} + y\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = 0.$$

This is an equation of the form $ax + by + c = 0$, and since the points (x_1, y_1) and (x_2, y_2) are distinct, at least one of a and b is not zero. Thus the equation is the equation of a line. The points (x_1, y_1) and (x_2, y_2) are on the line, because when the coordinates of one of the points are substituted for x and y, two rows of the matrix are equal and so the determinant is zero.

8. Expand along the first row to obtain

$$\begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 0 & 1 & m \end{vmatrix} = 1\begin{vmatrix} x_1 & y_1 \\ 1 & m \end{vmatrix} - x\begin{vmatrix} 1 & y_1 \\ 0 & m \end{vmatrix} + y\begin{vmatrix} 1 & x_1 \\ 0 & 1 \end{vmatrix} = 1(mx_1 - y_1) - x(m) + y(1) = 0.$$ This equation may be rewritten as $mx_1 - y_1 - mx + y = 0$, or $y - y_1 = m(x - x_1)$.

9. $\det T = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{vmatrix}$

$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix} = (b-a)(c-a)(c-b)$

10. Expanding along the first row will show that $f(t) = \det V = c_0 + c_1 t + c_2 t^2 + c_3 t^3$. By Exercise 9,

$$c_3 = -\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2) \neq 0$$

since x_1, x_2, and x_3 are distinct. Thus $f(t)$ is a cubic polynomial. The points $(x_1, 0)$, $(x_2, 0)$, and $(x_3, 0)$ are on the graph of f, since when any of x_1, x_2 or x_3 are substituted for t, the matrix has two equal rows and thus its determinant (which is $f(t)$) is zero. Thus $f(x_i) = 0$ for $i = 1, 2, 3$.

11. To tell if a quadrilateral determined by four points is a parallelogram, first translate one of the vertices to the origin. If we label the vertices of this new quadrilateral as $\mathbf{0}$, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, then they will be the vertices of a parallelogram if one of $\mathbf{v}_1$, $\mathbf{v}_2$, or $\mathbf{v}_3$ is the sum of the other two. In this example, subtract (1, 4) from each vertex to get a new parallelogram with vertices $\mathbf{0} = (0, 0)$, $\mathbf{v}_1 = (-2, 1)$, $\mathbf{v}_2 = (2, 5)$, and $\mathbf{v}_3 = (4, 4)$. Since $\mathbf{v}_2 = \mathbf{v}_3 + \mathbf{v}_1$, the quadrilateral is a parallelogram as stated. The translated parallelogram has the same area as the original, and is determined by the columns of $A = [\mathbf{v}_1 \quad \mathbf{v}_3] = \begin{bmatrix} -2 & 4 \\ 1 & 4 \end{bmatrix}$, so the area of the parallelogram is $|\det A| = |-12| = 12$.

12. A 2×2 matrix A is invertible if and only if the parallelogram determined by the columns of A has nonzero area.

13. By Theorem 8, $(\text{adj } A) \cdot \dfrac{1}{\det A} A = A^{-1}A = I$. By the Invertible Matrix Theorem, adj A is invertible and $(\text{adj } A)^{-1} = \dfrac{1}{\det A} A$.

14. a. Consider the matrix $A_k = \begin{bmatrix} A & O \\ O & I_k \end{bmatrix}$, where $1 \le k \le n$ and O is an appropriately sized zero matrix. We will show that $\det A_k = \det A$ for all $1 \le k \le n$ by mathematical induction.

First let $k = 1$. Expand along the last row to obtain

$\det A_1 = \det\begin{bmatrix} A & O \\ O & 1 \end{bmatrix} = (-1)^{(n+1)+(n+1)} \cdot 1 \cdot \det A = \det A.$

Now let $1 < k \le n$ and assume that $\det A_{k-1} = \det A$. Expand along the last row of A_k to obtain

$\det A_k = \det\begin{bmatrix} A & O \\ O & I_k \end{bmatrix} = (-1)^{(n+k)+(n+k)} \cdot 1 \cdot \det A_{k-1} = \det A_{k-1} = \det A.$ Thus we have proven the result, and the determinant of the matrix in question is $\det A$.

b. Consider the matrix $A_k = \begin{bmatrix} I_k & O \\ C_k & D \end{bmatrix}$, where $1 \le k \le n$, C_k is an $n \times k$ matrix and O is an appropriately sized zero matrix. We will show that $\det A_k = \det D$ for all $1 \le k \le n$ by mathematical induction.

First let $k = 1$. Expand along the first row to obtain

$$\det A_1 = \det\begin{bmatrix} 1 & O \\ C_1 & D \end{bmatrix} = (-1)^{1+1} \cdot 1 \cdot \det D = \det D.$$

Now let $1 < k \le n$ and assume that $\det A_{k-1} = \det D$. Expand along the first row of A_k to obtain

$\det A_k = \det\begin{bmatrix} I_k & O \\ C_k & D \end{bmatrix} = (-1)^{1+1} \cdot 1 \cdot \det A_{k-1} = \det A_{k-1} = \det D$. Thus we have proven the result, and the determinant of the matrix in question is $\det D$.

c. By combining parts a. and b., we have shown that

$$\det\begin{bmatrix} A & O \\ C & D \end{bmatrix} = \left(\det\begin{bmatrix} A & O \\ O & I \end{bmatrix}\right)\left(\det\begin{bmatrix} I & O \\ C & D \end{bmatrix}\right) = (\det A)(\det D).$$

From this result and Theorem 5, we have

$$\det\begin{bmatrix} A & B \\ O & D \end{bmatrix} = \det\begin{bmatrix} A & B \\ O & D \end{bmatrix}^T = \det\begin{bmatrix} A^T & O \\ B^T & D^T \end{bmatrix} = (\det A^T)(\det D^T) = (\det A)(\det D).$$

15. a. Compute the right side of the equation:

$$\begin{bmatrix} I & O \\ X & I \end{bmatrix}\begin{bmatrix} A & B \\ O & Y \end{bmatrix} = \begin{bmatrix} A & B \\ XA & XB+Y \end{bmatrix}$$

Set this equal to the left side of the equation:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ XA & XB+Y \end{bmatrix} \text{ so that } XA = C \quad XB + Y = D$$

Since $XA = C$ and A is invertible, $X = CA^{-1}$. Since $XB + Y = D$, $Y = D - XB = D - CA^{-1}B$. Thus by Exercise 14(c),

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det\begin{bmatrix} I & O \\ CA^{-1} & I \end{bmatrix} \det\begin{bmatrix} A & B \\ O & D - CA^{-1}B \end{bmatrix}$$

$$= (\det A)(\det(D - CA^{-1}B))$$

b. From part a.,

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = (\det A)(\det(D - CA^{-1}B)) = \det[A(D - CA^{-1}B)]$$

$$= \det[AD - ACA^{-1}B] = \det[AD - CAA^{-1}B]$$

$$= \det[AD - CB]$$

16. a. Doing the given operations does not change the determinant of A since the given operations are all row replacement operations. The resulting matrix is

$$\begin{bmatrix} a-b & -a+b & 0 & \dots & 0 \\ 0 & a-b & -a+b & \dots & 0 \\ 0 & 0 & a-b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{bmatrix}$$

b. Since column replacement operations are equivalent to row operations on A^T and $\det A^T = \det A$, the given operations do not change the determinant of the matrix. The resulting matrix is

$$\begin{bmatrix} a-b & 0 & 0 & \dots & 0 \\ 0 & a-b & 0 & \dots & 0 \\ 0 & 0 & a-b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & 2b & 3b & \dots & a+(n-1)b \end{bmatrix}$$

c. Since the preceding matrix is a triangular matrix with the same determinant as A,

$$\det A = (a-b)^{n-1}(a + (n-1)b).$$

17. First consider the case $n = 2$. In this case

$$\det B = \begin{vmatrix} a-b & b \\ 0 & a \end{vmatrix} = a(a-b), \det C = \begin{vmatrix} b & b \\ b & a \end{vmatrix} = ab - b^2,$$

so $\det A = \det B + \det C = a(a-b) + ab - b^2 = a^2 - b^2 = (a-b)(a+b) = (a-b)^{2-1}(a + (2-1)b)$, and the formula holds for $n = 2$.

Now assume that the formula holds for all $(k-1) \times (k-1)$ matrices, and let A, B, and C be $k \times k$ matrices. By a cofactor expansion along the first column,

$$\det B = (a-b)\begin{vmatrix} a & b & \dots & b \\ b & a & \dots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \dots & a \end{vmatrix} = (a-b)(a-b)^{k-2}(a + (k-2)b) = (a-b)^{k-1}(a + (k-2)b)$$

since the matrix in the above formula is a $(k-1) \times (k-1)$ matrix. We can perform a series of row operations on C to "zero out" below the first pivot, and produce the following matrix whose determinant is $\det C$:

$$\begin{bmatrix} b & b & \dots & b \\ 0 & a-b & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a-b \end{bmatrix}.$$

Since this is a triangular matrix, we have found that $\det C = b(a-b)^{k-1}$. Thus

$$\det A = \det B + \det C = (a-b)^{k-1}(a + (k-2)b) + b(a-b)^{k-1} = (a-b)^{k-1}(a + (k-1)b),$$

which is what was to be shown. Thus the formula has been proven by mathematical induction.

18. **[M]** Since the first matrix has $a = 3$, $b = 8$, and $n = 4$, its determinant is $(3-8)^{4-1}(3 + (4-1)8) = (-5)^3(3+24) = (-125)(27) = -3375$. Since the second matrix has $a = 8$, $b = 3$, and $n = 5$, its determinant is $(8-3)^{5-1}(8 + (5-1)3) = (5)^4(8+12) = (625)(20) = 12{,}500$.

19. **[M]** We find that

$$\begin{vmatrix}1&1&1\\1&2&2\\1&2&3\end{vmatrix}=1,\begin{vmatrix}1&1&1&1\\1&2&2&2\\1&2&3&3\\1&2&3&4\end{vmatrix}=1,\begin{vmatrix}1&1&1&1&1\\1&2&2&2&2\\1&2&3&3&3\\1&2&3&4&4\\1&2&3&4&5\end{vmatrix}=1.$$

Our conjecture then is that

$$\begin{vmatrix}1&1&1&\dots&1\\1&2&2&\dots&2\\1&2&3&\dots&3\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&2&3&\dots&n\end{vmatrix}=1.$$

To show this, consider using row replacement operations to "zero out" below the first pivot. The resulting matrix is

$$\begin{bmatrix}1&1&1&\dots&1\\0&1&1&\dots&1\\0&1&2&\dots&2\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&1&2&\dots&n-1\end{bmatrix}.$$

Now use row replacement operations to "zero out" below the second pivot, and so on. The final matrix which results from this process is

$$\begin{bmatrix}1&1&1&\dots&1\\0&1&1&\dots&1\\0&0&1&\dots&1\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\dots&1\end{bmatrix},$$

which is an upper triangular matrix with determinant 1.

20. **[M]** We find that

$$\begin{vmatrix}1&1&1\\1&3&3\\1&3&6\end{vmatrix}=6,\begin{vmatrix}1&1&1&1\\1&3&3&3\\1&3&6&6\\1&3&6&9\end{vmatrix}=18,\begin{vmatrix}1&1&1&1&1\\1&3&3&3&3\\1&3&6&6&6\\1&3&6&9&9\\1&3&6&9&12\end{vmatrix}=54.$$

Our conjecture then is that

$$\begin{vmatrix}1&1&1&\dots&1\\1&3&3&\dots&3\\1&3&6&\dots&6\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&3&6&\dots&3(n-1)\end{vmatrix}=2\cdot 3^{n-2}.$$

To show this, consider using row replacement operations to "zero out" below the first pivot. The resulting matrix is

$$\begin{bmatrix}1&1&1&\dots&1\\0&2&2&\dots&2\\0&2&5&\dots&5\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&2&5&\dots&3(n-1)-1\end{bmatrix}.$$

Now use row replacement operations to "zero out" below the second pivot. The matrix which results from this process is

$$\begin{bmatrix}1&1&1&1&1&1&\dots&1\\0&2&2&2&2&2&\dots&2\\0&0&3&3&3&3&\dots&3\\0&0&3&6&6&6&\dots&6\\0&0&3&6&9&9&\dots&9\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&3&6&9&12&\dots&3(n-2)\end{bmatrix}.$$

This matrix has the same determinant as the original matrix, and is recognizable as a block matrix of the form

$$\begin{bmatrix}A&B\\O&D\end{bmatrix},$$

where

$$A=\begin{bmatrix}1&1\\0&2\end{bmatrix}\text{ and }D=\begin{bmatrix}3&3&3&3&\dots&3\\3&6&6&6&\dots&6\\3&6&9&9&\dots&9\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\3&6&9&12&\dots&3(n-2)\end{bmatrix}=3\begin{bmatrix}1&1&1&1&\dots&1\\1&2&2&2&\dots&2\\1&2&3&3&\dots&3\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\1&2&3&4&\dots&n-2\end{bmatrix}.$$

As in Exercise 14(c), the determinant of the matrix $\begin{bmatrix}A&B\\O&D\end{bmatrix}$ is (det A)(det D) = 2 det D.

Since D is an $(n-2)\times(n-2)$ matrix,