45. **[M]** Answers will vary. For 4×4 matrices, the conclusions should be that $\det A^T = \det A$, $\det(-A) = \det A$, $\det(2A) = 16\det A$, and $\det(10A) = 10^4 \det A$. For 5×5 matrices, the conclusions should be that $\det A^T = \det A$, $\det(-A) = -\det A$, $\det(2A) = 32\det A$, and $\det(10A) = 10^5 \det A$. For 6×6 matrices, the conclusions should be that $\det A^T = \det A$, $\det(-A) = \det A$, $\det(2A) = 64\det A$, and $\det(10A) = 10^6 \det A$.

46. **[M]** Answers will vary. The conclusion should be that $\det A^{-1} = 1/\det A$.

3.2 SOLUTIONS

Notes: This section presents the main properties of the determinant, including the effects of row operations on the determinant of a matrix. These properties are first studied by examples in Exercises 1–20. The properties are treated in a more theoretical manner in later exercises. An efficient method for computing the determinant using row reduction and selective cofactor expansion is presented in this section and used in Exercises 11–14. Theorems 4 and 6 are used extensively in Chapter 5. The linearity property of the determinant studied in the text is optional, but is used in more advanced courses.

1. Rows 1 and 2 are interchanged, so the determinant changes sign (Theorem 3b.).

2. The constant 2 may be factored out of the Row 1 (Theorem 3c.).

3. The row replacement operation does not change the determinant (Theorem 3a.).

4. The row replacement operation does not change the determinant (Theorem 3a.).

5. $$\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 3 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{vmatrix} = 3$$

6. $$\begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -3 \\ 0 & -18 & 12 \\ 0 & 3 & -1 \end{vmatrix} = 6\begin{vmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 3 & -1 \end{vmatrix} = 6\begin{vmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{vmatrix} = (6)(-3) = -18$$

7. $$\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & -4 & 2 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 30 & 27 \\ 0 & 0 & 30 & 27 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 30 & 27 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

8. $$\begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 5 \\ 0 & 2 & 4 & -10 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

9. $$\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 1 & 5 & 5 \\ 0 & 2 & 7 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & -5 \end{vmatrix} = -\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & -3 & -5 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -(-3) = 3$$

10. $$\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ -2 & -6 & 2 & 3 & 9 \\ 3 & 7 & -3 & 8 & -7 \\ 3 & 5 & 5 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & -2 & 0 & 8 & -1 \\ 0 & -4 & 8 & 2 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & -4 & 7 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} =$$

$$-\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ 0 & 0 & -4 & 7 & -7 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = -(-24) = 24$$

11. First use a row replacement to create zeros in the second column, and then expand down the second column:

$$\begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 0 & 0 & 2 & 1 \end{vmatrix} = -5\begin{vmatrix} 3 & 1 & -3 \\ -6 & -4 & 9 \\ 0 & 2 & 1 \end{vmatrix}$$

Now use a row replacement to create zeros in the first column, and then expand down the first column:

$$-5\begin{vmatrix} 3 & 1 & -3 \\ -6 & -4 & 9 \\ 0 & 2 & 1 \end{vmatrix} = -5\begin{vmatrix} 3 & 1 & -3 \\ 0 & -2 & 3 \\ 0 & 2 & 1 \end{vmatrix} = (-5)(3)\begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix} = (-5)(3)(-8) = 120$$

12. First use a row replacement to create zeros in the fourth column, and then expand down the fourth column:

$$\begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ -3 & 0 & -2 & 0 \\ 4 & 2 & 4 & 3 \end{vmatrix} = 3\begin{vmatrix} -1 & 2 & 3 \\ 3 & 4 & 3 \\ -3 & 0 & -2 \end{vmatrix}$$

Now use a row replacement to create zeros in the first column, and then expand down the first column: $$3\begin{vmatrix} -1 & 2 & 3 \\ 3 & 4 & 3 \\ -3 & 0 & -2 \end{vmatrix} = 3\begin{vmatrix} -1 & 2 & 3 \\ 0 & 10 & 12 \\ 0 & -6 & -11 \end{vmatrix} = 3(-1)\begin{vmatrix} 10 & 12 \\ -6 & -11 \end{vmatrix} = 3(-1)(-38) = 114$$

13. First use a row replacement to create zeros in the fourth column, and then expand down the fourth column:

$$\begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 5 & 4 & 1 \\ 0 & -3 & -2 & 0 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix} = -1\begin{vmatrix} 0 & -3 & -2 \\ 6 & -2 & -4 \\ -6 & 7 & 7 \end{vmatrix}$$

Now use a row replacement to create zeros in the first column, and then expand down the first column: $-1\begin{vmatrix} 0 & -3 & -2 \\ 6 & -2 & -4 \\ -6 & 7 & 7 \end{vmatrix} = -1\begin{vmatrix} 0 & -3 & -2 \\ 6 & -2 & -4 \\ 0 & 5 & 3 \end{vmatrix} = (-1)(-6)\begin{vmatrix} -3 & -2 \\ 5 & 3 \end{vmatrix} = (-1)(-6)(1) = 6$

14. First use a row replacement to create zeros in the third column, and then expand down the third column:

$$\begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{vmatrix} = \begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -9 & 0 & 0 & 0 \\ 3 & -4 & 0 & 4 \end{vmatrix} = 1\begin{vmatrix} 1 & 3 & -3 \\ -9 & 0 & 0 \\ 3 & -4 & 4 \end{vmatrix}$$

Now expand along the second row:

$$1\begin{vmatrix} 1 & 3 & -3 \\ -9 & 0 & 0 \\ 3 & -4 & 4 \end{vmatrix} = 1(-(-9))\begin{vmatrix} 3 & -3 \\ -4 & 4 \end{vmatrix} = (1)(9)(0) = 0$$

15. $\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix} = 5\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5(7) = 35$

16. $\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix} = 3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3(7) = 21$

17. $\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -7$

18. $\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -\left(-\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\right) = -(-7) = 7$

19. $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2(7) = 14$

20. $\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$

21. Since $\begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = -1 \neq 0$, the matrix is invertible.

22. Since $\begin{vmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix} = 0$, the matrix is not invertible.

23. Since $\begin{vmatrix} 2 & 0 & 0 & 8 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{vmatrix} = 0$, the matrix is not invertible.

24. Since $\begin{vmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ -7 & 2 & 6 \end{vmatrix} = 11 \neq 0$, the columns of the matrix form a linearly independent set.

25. Since $\begin{vmatrix} 7 & -8 & 7 \\ -4 & 5 & 0 \\ -6 & 7 & -5 \end{vmatrix} = -1 \neq 0$, the columns of the matrix form a linearly independent set.

26. Since $\begin{vmatrix} 3 & 2 & -2 & 0 \\ 5 & -6 & -1 & 0 \\ -6 & 0 & 3 & 0 \\ 4 & 7 & 0 & -3 \end{vmatrix} = 0$, the columns of the matrix form a linearly dependent set.

27. **a.** True. See Theorem 3.
b. True. See the paragraph following Example 2.
c. True. See the paragraph following Theorem 4.
d. False. See the warning following Example 5.

28. **a.** True. See Theorem 3.
b. False. See the paragraphs following Example 2.
c. False. See Example 3.
d. False. See Theorem 5.

29. By Theorem 6, $\det B^5 = (\det B)^5 = (-2)^5 = -32$.

30. Suppose the two rows of a square matrix A are equal. By swapping these two rows, the matrix A is not changed so its determinant should not change. But since swapping rows changes the sign of the determinant, det A = – det A. This is only possible if det A = 0. The same may be proven true for columns by applying the above result to A^T and using Theorem 5.

31. By Theorem 6, $(\det A)(\det A^{-1}) = \det I = 1$, so $\det A^{-1} = 1/\det A$.

32. By factoring an r out of each of the n rows, $\det(rA) = r^n \det A$.

33. By Theorem 6, det AB = (det A)(det B) = (det B)(det A) = det BA.

34. By Theorem 6 and Exercise 31,

$$\det(PAP^{-1}) = (\det P)(\det A)(\det P^{-1}) = (\det P)(\det P^{-1})(\det A)$$
$$= (\det P)\left(\frac{1}{\det P}\right)(\det A) = 1 \det A$$
$$= \det A$$

35. By Theorem 6 and Theorem 5, $\det U^TU = (\det U^T)(\det U) = (\det U)^2$. Since $U^TU = I$, $\det U^TU = \det I = 1$, so $(\det U)^2 = 1$. Thus det $U = \pm 1$.

36. By Theorem 6 $\det A^4 = (\det A)^4$. Since $\det A^4 = 0$, then $(\det A)^4 = 0$. Thus det A = 0, and A is not invertible by Theorem 4.

37. One may compute using Theorem 2 that det A = 3 and det B = 8, while $AB = \begin{bmatrix} 6 & 0 \\ 17 & 4 \end{bmatrix}$. Thus det $AB = 24 = 3 \times 8$ = (det A)(det B).

38. One may compute that det A = 0 and det B = –2, while $AB = \begin{bmatrix} 6 & 0 \\ -2 & 0 \end{bmatrix}$. Thus det $AB = 0 = 0 \times -2$ = (det A)(det B).

39. **a.** By Theorem 6, det AB = (det A)(det B) = $4 \times -3 = -12$.

b. By Exercise 32, $\det 5A = 5^3 \det A = 125 \times 4 = 500$.

c. By Theorem 5, $\det B^T = \det B = -3$.

d. By Exercise 31, $\det A^{-1} = 1/\det A = 1/4$.

e. By Theorem 6, $\det A^3 = (\det A)^3 = 4^3 = 64$.

40. **a.** By Theorem 6, det AB = (det A)(det B) = $-1 \times 2 = -2$.

b. By Theorem 6, $\det B^5 = (\det B)^5 = 2^5 = 32$.

c. By Exercise 32, $\det 2A = 2^4 \det A = 16 \times -1 = -16$.

d. By Theorems 5 and 6, $\det A^TA = (\det A^T)(\det A) = (\det A)(\det A) = -1 \times -1 = 1$.

e. By Theorem 6 and Exercise 31,
$\det B^{-1}AB = (\det B^{-1})(\det A)(\det B) = (1/\det B)(\det A)(\det B) = \det A = -1$.

41. det $A = (a + e)d - c(b + f) = ad + ed - bc - cf = (ad - bc) + (ed - cf)$ = det B + det C.

42. $\det(A+B) = \begin{vmatrix} 1+a & b \\ c & 1+d \end{vmatrix} = (1+a)(1+d) - cb = 1 + a + d + ad - cb = \det A + a + d + \det B$, so det $(A + B)$ = det A + det B if and only if $a + d = 0$.

43. Compute det A by using a cofactor expansion down the third column:

$$\det A = (u_1 + v_1)\det A_{13} - (u_2 + v_2)\det A_{23} + (u_3 + v_3)\det A_{33}$$
$$= u_1 \det A_{13} - u_2 \det A_{23} + u_3 \det A_{33} + v_1 \det A_{13} - v_2 \det A_{23} + v_3 \det A_{33}$$
$$= \det B + \det C$$

44. By Theorem 5, $\det AE = \det(AE)^T$. Since $(AE)^T = E^TA^T$, $\det AE = \det(E^TA^T)$. Now E^T is itself an elementary matrix, so by the proof of Theorem 3, $\det(E^TA^T) = (\det E^T)(\det A^T)$. Thus it is true that $\det AE = (\det E^T)(\det A^T)$, and by applying Theorem 5, det AE = (det E)(det A).

45. **[M]** Answers will vary, but will show that $\det A^TA$ always equals 0 while $\det AA^T$ should seldom be zero. To see why A^TA - should not be invertible (and thus $\det A^TA = 0$), let A be a matrix with more columns than rows. Then the columns of A must be linearly dependent, so the equation $A\mathbf{x} = \mathbf{0}$ must have a non-trivial solution $\mathbf{x}$. Thus $(A^TA)\mathbf{x} = A^T(A\mathbf{x}) = A^T\mathbf{0} = \mathbf{0}$, and the equation $(A^TA)\mathbf{x} = \mathbf{0}$ has a
non-trivial solution. Since A^TA is a square matrix, the Invertible Matrix Theorem now says that A^TA is not invertible. Notice that the same argument will not work in general for AA^T, since A^T has more rows than columns, so its columns are not automatically linearly dependent.

46. **[M]** One may compute for this matrix that det $A = -4008$ and cond $A \approx 16.3$. Note that this is the ℓ_2 condition number, which is used in Section 2.3. Since det $A \neq 0$, it is invertible and

$$A^{-1} = -\frac{1}{4008}\begin{bmatrix} -837 & -181 & -207 & 297 \\ -750 & -574 & 30 & 654 \\ 171 & 195 & -87 & -1095 \\ 21 & -187 & -81 & 639 \end{bmatrix}$$

The determinant is very sensitive to scaling, as $\det 10A = 10^4 \det A = -40{,}080{,}000$ and $\det 0.1A = (0.1)^4 \det A = -0.4008$. The condition number is not changed at all by scaling: cond(10A) = cond(0.1A) = cond $A \approx 16.3$. When $A = I_4$, det A=1 and cond A = 1. As before the determinant is sensitive to scaling: $\det 10A = 10^4 \det A = 10{,}000$ and $\det 0.1A = (0.1)^4 \det A = 0.0001$. Yet the condition number is not changed by scaling: cond(10A) = cond(0.1A) = cond A = 1.