Math 260 Homework 2.6

$$U=\begin{bmatrix}4 & -1 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 3.75 & -.25 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 3.7333 & -1.0667 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 3.4286 & -.2857 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 3.7083 & -1.0833 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 3.3919 & -.2921 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 & 3.7052 & -1.0861\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.3868\end{bmatrix}$$

b. The result of solving $L\mathbf{y}=\mathbf{b}$ and then $U\mathbf{x}=\mathbf{y}$ is

$\mathbf{x}$ = (27.1292, 19.2344, 29.2823, 19.8086, 30.1914, 20.7177, 30.7656, 22.8708)

c. $$A^{-1}=\begin{bmatrix}.2953 & .0866 & .0945 & .0509 & .0318 & .0227 & .0010 & .0082\\ .0866 & .2953 & .0509 & .0945 & .0227 & .0318 & .0082 & .0100\\ .0945 & .0509 & .3271 & .1093 & .1045 & .0591 & .0318 & .0227\\ .0509 & .0945 & .1093 & .3271 & .0591 & .1045 & .0227 & .0318\\ .0318 & .0227 & .1045 & .0591 & .3271 & .1093 & .0945 & .0509\\ .0227 & .0318 & .0591 & .1045 & .1093 & .3271 & .0509 & .0945\\ .0010 & .0082 & .0318 & .0227 & .0945 & .0509 & .2953 & .0866\\ .0082 & .0100 & .0227 & .0318 & .0509 & .0945 & .0866 & .2953\end{bmatrix}$$

32. [M] $A=\begin{bmatrix}3 & -1 & 0 & 0\\ -1 & 3 & -1 & 0\\ 0 & -1 & 3 & -1\\ 0 & 0 & -1 & 3\end{bmatrix}$. The commands shown for Exercise 31, but modified for 4×4 matrices, produce

$$L=\begin{bmatrix}1 & 0 & 0 & 0\\ -\frac{1}{3} & 1 & 0 & 0\\ 0 & -\frac{3}{8} & 1 & 0\\ 0 & 0 & -\frac{8}{21} & 1\end{bmatrix}$$

$$U=\begin{bmatrix}3 & -1 & 0 & 0\\ 0 & \frac{8}{3} & -1 & 0\\ 0 & 0 & \frac{21}{8} & -1\\ 0 & 0 & 0 & \frac{55}{21}\end{bmatrix}$$

b. Let $\mathbf{s}_{k+1}$ be the solution of $L\mathbf{s}_{k+1}=\mathbf{t}_k$ for k = 0, 1, 2, …. Then $\mathbf{t}_{k+1}$ is the solution of $U\mathbf{t}_{k+1}=\mathbf{s}_{k+1}$ for k = 0, 1, 2, …. The results are

$$\mathbf{s}_1=\begin{bmatrix}10.0000\\18.3333\\21.8750\\18.3333\end{bmatrix},\mathbf{t}_1=\begin{bmatrix}7.0000\\11.0000\\11.0000\\7.0000\end{bmatrix},\mathbf{s}_2=\begin{bmatrix}7.0000\\13.3333\\16.0000\\13.0952\end{bmatrix},\mathbf{t}_2=\begin{bmatrix}5.0000\\8.0000\\8.0000\\5.0000\end{bmatrix},$$

$$\mathbf{s}_3=\begin{bmatrix}5.0000\\9.6667\\11.6250\\9.4286\end{bmatrix},\mathbf{t}_3=\begin{bmatrix}3.6000\\5.8000\\5.8000\\3.6000\end{bmatrix},\mathbf{s}_4=\begin{bmatrix}3.6000\\7.0000\\8.4250\\6.8095\end{bmatrix},\mathbf{t}_4=\begin{bmatrix}2.6000\\4.2000\\4.2000\\2.6000\end{bmatrix}.$$

2.6 SOLUTIONS

Notes: This section is independent of Section 1.10. The material here makes a good backdrop for the series expansion of $(I–C)^{-1}$ because this formula is actually used in some practical economic work. Exercise 8 gives an interpretation to entries of an inverse matrix that could be stated without the economic context.

1. The answer to this exercise will depend upon the order in which the student chooses to list the sectors. The important fact to remember is that each column is the unit consumption vector for the appropriate sector. If we order the sectors manufacturing, agriculture, and services, then the consumption matrix is

$$C=\begin{bmatrix}.10 & .60 & .60\\ .30 & .20 & 0\\ .30 & .10 & .10\end{bmatrix}$$

The intermediate demands created by the production vector $\mathbf{x}$ are given by $C\mathbf{x}$. Thus in this case the intermediate demand is

$$C\mathbf{x}=\begin{bmatrix}.10 & .60 & .60\\ .30 & .20 & .00\\ .30 & .10 & .10\end{bmatrix}\begin{bmatrix}0\\100\\0\end{bmatrix}=\begin{bmatrix}60\\20\\10\end{bmatrix}$$

2. Solve the equation $\mathbf{x}=C\mathbf{x}+\mathbf{d}$ for $\mathbf{d}$:

$$\mathbf{d}=\mathbf{x}-C\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}-\begin{bmatrix}.10 & .60 & .60\\ .30 & .20 & .00\\ .30 & .10 & .10\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}.9x_1 & -.6x_2 & -.6x_3\\ -.3x_1 & +.8x_2 & \\ -.3x_1 & -.1x_2 & +.9x_3\end{bmatrix}=\begin{bmatrix}0\\20\\0\end{bmatrix}$$

This system of equations has the augmented matrix

$$\begin{bmatrix}.90 & -.60 & -.60 & 0\\ -.30 & .80 & .00 & 20\\ -.30 & -.10 & .90 & 0\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 0 & 37.03\\ 0 & 1 & 0 & 38.89\\ 0 & 0 & 1 & 16.67\end{bmatrix},\text{ so }\mathbf{x}=\begin{bmatrix}37.03\\38.89\\16.67\end{bmatrix}.$$

3. Solving as in Exercise 2:

$$\mathbf{d}=\mathbf{x}-C\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}-\begin{bmatrix}.10&.60&.60\\.30&.20&.00\\.30&.10&.10\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}.9x_1&-.6x_2&-.6x_3\\-.3x_1&+.8x_2&\\-.3x_1&-.1x_2&+.9x_3\end{bmatrix}=\begin{bmatrix}20\\0\\0\end{bmatrix}$$

This system of equations has the augmented matrix

$$\begin{bmatrix}.90&-.60&-.60&20\\-.30&.80&.00&0\\-.30&-.10&.90&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&44.44\\0&1&0&16.67\\0&0&1&16.67\end{bmatrix},\text{ so }\mathbf{x}=\begin{bmatrix}44.44\\16.67\\16.67\end{bmatrix}.$$

4. Solving as in Exercise 2:

$$\mathbf{d}=\mathbf{x}-C\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}-\begin{bmatrix}.10&.60&.60\\.30&.20&.00\\.30&.10&.10\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}.9x_1&-.6x_2&-.6x_3\\-.3x_1&+.8x_2&\\-.3x_1&-.1x_2&+.9x_3\end{bmatrix}=\begin{bmatrix}20\\20\\0\end{bmatrix}$$

This system of equations has the augmented matrix

$$\begin{bmatrix}.90&-.60&-.60&20\\-.30&.80&.00&20\\-.30&-.10&.90&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&81.48\\0&1&0&55.56\\0&0&1&33.33\end{bmatrix},\text{ so }\mathbf{x}=\begin{bmatrix}81.48\\55.56\\33.33\end{bmatrix}.$$

Note: Exercises 2–4 may be used by students to discover the linearity of the Leontief model.

5. $\mathbf{x}=(I-C)^{-1}\mathbf{d}=\begin{bmatrix}1&-.5\\-.6&.8\end{bmatrix}^{-1}\begin{bmatrix}50\\30\end{bmatrix}=\begin{bmatrix}1.6&1\\1.2&2\end{bmatrix}\begin{bmatrix}50\\30\end{bmatrix}=\begin{bmatrix}110\\120\end{bmatrix}$

6. $\mathbf{x}=(I-C)^{-1}\mathbf{d}=\begin{bmatrix}.8&-.5\\-.6&.9\end{bmatrix}^{-1}\begin{bmatrix}16\\12\end{bmatrix}=\dfrac{1}{(.72-.30)}\begin{bmatrix}.9&.5\\.6&.8\end{bmatrix}\begin{bmatrix}16\\12\end{bmatrix}=\begin{bmatrix}48.57\\45.71\end{bmatrix}$

7. a. From Exercise 5,

$$(I-C)^{-1}=\begin{bmatrix}1.6&1\\1.2&2\end{bmatrix}$$

so

$$\mathbf{x}_1=(I-C)^{-1}\mathbf{d}_1=\begin{bmatrix}1.6&1\\1.2&2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1.6\\1.2\end{bmatrix}$$

which is the first column of $(I-C)^{-1}$.

b. $\mathbf{x}_2=(I-C)^{-1}\mathbf{d}_2=\begin{bmatrix}1.6&1\\1.2&2\end{bmatrix}\begin{bmatrix}51\\30\end{bmatrix}=\begin{bmatrix}111.6\\121.2\end{bmatrix}$

c. From Exercise 5, the production **x** corressponding to $\mathbf{d}=\begin{bmatrix}50\\30\end{bmatrix}$ is $\mathbf{x}=\begin{bmatrix}110\\120\end{bmatrix}$.

Note that $\mathbf{d}_2=\mathbf{d}+\mathbf{d}_1$. Thus

$$\begin{aligned}\mathbf{x}_2&=(I-C)^{-1}\mathbf{d}_2\\&=(I-C)^{-1}(\mathbf{d}+\mathbf{d}_1)\\&=(I-C)^{-1}\mathbf{d}+(I-C)^{-1}\mathbf{d}_1\\&=\mathbf{x}+\mathbf{x}_1\end{aligned}$$

8. a. Given $(I-C)\mathbf{x}=\mathbf{d}$ and $(I-C)\Delta\mathbf{x}=\Delta\mathbf{d}$,

$$(I-C)(\mathbf{x}+\Delta\mathbf{x})=(I-C)\mathbf{x}+(I-C)\Delta\mathbf{x}=\mathbf{d}+\Delta\mathbf{d}$$

Thus $\mathbf{x}+\Delta\mathbf{x}$ is the production level corresponding to a demand of $\mathbf{d}+\Delta\mathbf{d}$.

b. Since $\Delta\mathbf{x}=(I-C)^{-1}\Delta\mathbf{d}$ and $\Delta\mathbf{d}$ is the first column of I, $\Delta\mathbf{x}$ will be the first column of $(I-C)^{-1}$.

9. In this case

$$I-C=\begin{bmatrix}.8&-.2&.0\\-.3&.9&-.3\\-.1&.0&.8\end{bmatrix}$$

Row reduce $[\,I-C\ \ \mathbf{d}\,]$ to find

$$\begin{bmatrix}.8&-.2&.0&40.0\\-.3&.9&-.3&60.0\\-.1&.0&.8&80.0\end{bmatrix}\sim\begin{bmatrix}1&0&0&82.8\\0&1&0&131.0\\0&0&1&110.3\end{bmatrix}$$

So **x** = (82.8, 131.0, 110.3).

10. From Exercise 8, the (i, j) entry in $(I-C)^{-1}$ corresponds to the effect on production of sector i when the final demand for the output of sector j increases by one unit. Since these entries are all positive, an increase in the final demand for any sector will cause the production of all sectors to increase. Thus an increase in the demand for any sector will lead to an increase in the demand for all sectors.

11. (Solution in *study Guide*) Following the hint in the text, compute $\mathbf{p}^T\mathbf{x}$ in two ways. First, take the transpose of both sides of the price equation, $\mathbf{p}=C^T\mathbf{p}+\mathbf{v}$, to obtain

$$\mathbf{p}^T=(C^T\mathbf{p}+\mathbf{v})^T=(C^T\mathbf{p})^T+\mathbf{v}^T=\mathbf{p}^TC+\mathbf{v}^T$$

and right-multiply by **x** to get

$$\mathbf{p}^T\mathbf{x}=(\mathbf{p}^TC+\mathbf{v}^T)\mathbf{x}=\mathbf{p}^TC\mathbf{x}+\mathbf{v}^T\mathbf{x}$$

Another way to compute $\mathbf{p}^T\mathbf{x}$ starts with the production equation $\mathbf{x}=C\mathbf{x}+\mathbf{d}$. Left multiply by $\mathbf{p}^T$ to get

$$\mathbf{p}^T\mathbf{x}=\mathbf{p}^T(C\mathbf{x}+\mathbf{d})=\mathbf{p}^TC\mathbf{x}+\mathbf{p}^T\mathbf{d}$$

The two expression for $\mathbf{p}^T\mathbf{x}$ show that

$$\mathbf{p}^TC\mathbf{x}+\mathbf{v}^T\mathbf{x}=\mathbf{p}^TC\mathbf{x}+\mathbf{p}^T\mathbf{d}$$

so $\mathbf{v}^T\mathbf{x}=\mathbf{p}^T\mathbf{d}$. The *Study Guide* also provides a slightly different solution.

12. Since

$$D_{m+1} = I + C + C^2 + ... + C^{m+1} = I + C(I + C + ... + C^m) = I + CD_m$$

D_{m+1} may be found iteratively by $D_{m+1} = I + CD_m$.

13. [M] The matrix $I - C$ is

$$\begin{bmatrix} 0.8412 & -0.0064 & -0.0025 & -0.0304 & -0.0014 & -0.0083 & -0.1594 \\ -0.0057 & 0.7355 & -0.0436 & -0.0099 & -0.0083 & -0.0201 & -0.3413 \\ -0.0264 & -0.1506 & 0.6443 & -0.0139 & -0.0142 & -0.0070 & -0.0236 \\ -0.3299 & -0.0565 & -0.0495 & 0.6364 & -0.0204 & -0.0483 & -0.0649 \\ -0.0089 & -0.0081 & -0.0333 & -0.0295 & 0.6588 & -0.0237 & -0.0020 \\ -0.1190 & -0.0901 & -0.0996 & -0.1260 & -0.1722 & 0.7632 & -0.3369 \\ -0.0063 & -0.0126 & -0.0196 & -0.0098 & -0.0064 & -0.0132 & 0.9988 \end{bmatrix}$$

so the augmented matrix $[\,I - C \;\; \mathbf{d}\,]$ may be row reduced to find

$$\begin{bmatrix} 0.8412 & -0.0064 & -0.0025 & -0.0304 & -0.0014 & -0.0083 & -0.1594 & 74000 \\ -0.0057 & 0.7355 & -0.0436 & -0.0099 & -0.0083 & -0.0201 & -0.3413 & 56000 \\ -0.0264 & -0.1506 & 0.6443 & -0.0139 & -0.0142 & -0.0070 & -0.0236 & 10500 \\ -0.3299 & -0.0565 & -0.0495 & 0.6364 & -0.0204 & -0.0483 & -0.0649 & 25000 \\ -0.0089 & -0.0081 & -0.0333 & -0.0295 & 0.6588 & -0.0237 & -0.0020 & 17500 \\ -0.1190 & -0.0901 & -0.0996 & -0.1260 & -0.1722 & 0.7632 & -0.3369 & 196000 \\ -0.0063 & -0.0126 & -0.0196 & -0.0098 & -0.0064 & -0.0132 & 0.9988 & 5000 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 99576 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 97703 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 51231 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 131570 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 49488 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 329554 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 13835 \end{bmatrix}$$

so **x** = (99576, 97703, 51231, 131570, 49488, 329554, 13835). Since the entries in **d** seem to be accurate to the nearest thousand, a more realistic answer would be **x** = (100000, 98000, 51000, 132000, 49000, 330000, 14000).

14. [M] The augmented matrix $[\,I - C \;\; \mathbf{d}\,]$ in this case may be row reduced to find

$$\begin{bmatrix} 0.8412 & -0.0064 & -0.0025 & -0.0304 & -0.0014 & -0.0083 & -0.1594 & 99640 \\ -0.0057 & 0.7355 & -0.0436 & -0.0099 & -0.0083 & -0.0201 & -0.3413 & 75548 \\ -0.0264 & -0.1506 & 0.6443 & -0.0139 & -0.0142 & -0.0070 & -0.0236 & 14444 \\ -0.3299 & -0.0565 & -0.0495 & 0.6364 & -0.0204 & -0.0483 & -0.0649 & 33501 \\ -0.0089 & -0.0081 & -0.0333 & -0.0295 & 0.6588 & -0.0237 & -0.0020 & 23527 \\ -0.1190 & -0.0901 & -0.0996 & -0.1260 & -0.1722 & 0.7632 & -0.3369 & 263985 \\ -0.0063 & -0.0126 & -0.0196 & -0.0098 & -0.0064 & -0.0132 & 0.9988 & 6526 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 134034 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 131687 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 69472 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 176912 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 66596 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 443773 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 18431 \end{bmatrix}$$

so **x** = (134034, 131687, 69472, 176912, 66596, 443773, 18431). To the nearest thousand, **x** = (134000, 132000, 69000, 177000, 67000, 444000, 18000).

15. [M] Here are the iterations rounded to the nearest tenth:

$\mathbf{x}^{(0)} = (74000.0, 56000.0, 10500.0, 25000.0, 17500.0, 196000.0, 5000.0)$

$\mathbf{x}^{(1)} = (89344.2, 77730.5, 26708.1, 72334.7, 30325.6, 265158.2, 9327.8)$

$\mathbf{x}^{(2)} = (94681.2, 87714.5, 37577.3, 100520.5, 38598.0, 296563.8, 11480.0)$

$\mathbf{x}^{(3)} = (97091.9, 92573.1, 43867.8, 115457.0, 43491.0, 312319.0, 12598.8)$

$\mathbf{x}^{(4)} = (98291.6, 95033.2, 47314.5, 123202.5, 46247.0, 320502.4, 13185.5)$

$\mathbf{x}^{(5)} = (98907.2, 96305.3, 49160.6, 127213.7, 47756.4, 324796.1, 13493.8)$

$\mathbf{x}^{(6)} = (99226.6, 96969.6, 50139.6, 129296.7, 48569.3, 327053.8, 13655.9)$

$\mathbf{x}^{(7)} = (99393.1, 97317.8, 50656.4, 130381.6, 49002.8, 328240.9, 13741.1)$

$\mathbf{x}^{(8)} = (99480.0, 97500.7, 50928.7, 130948.0, 49232.5, 328864.7, 13785.9)$

$\mathbf{x}^{(9)} = (99525.5, 97596.8, 51071.9, 131244.1, 49353.8, 329192.3, 13809.4)$

$\mathbf{x}^{(10)} = (99549.4, 97647.2, 51147.2, 131399.2, 49417.7, 329364.4, 13821.7)$

$\mathbf{x}^{(11)} = (99561.9, 97673.7, 51186.8, 131480.4, 49451.3, 329454.7, 13828.2)$

$\mathbf{x}^{(12)} = (99568.4, 97687.6, 51207.5, 131523.0, 49469.0, 329502.1, 13831.6)$

so $\mathbf{x}^{(12)}$ is the first vector whose entries are accurate to the nearest thousand. The calculation of $\mathbf{x}^{(12)}$ takes about 1260 flops, while the row reduction above takes about 550 flops. If C is larger than 20×20, then fewer flops are required to compute $\mathbf{x}^{(12)}$ by iteration than by row reduction. The advantage of the iterative method increases with the size of C. The matrix C also becomes more sparse for larger models, so fewer iterations are needed for good accuracy.

2.7 SOLUTIONS

Notes: The content of this section seems to have universal appeal with students. It also provides practice with composition of linear transformations. The case study for Chapter 2 concerns computer graphics – see this case study (available as a project on the website) for more examples of computer graphics in action. The *Study Guide* encourages the student to examine the book by Foley referenced in the text. This section could form the beginning of an independent study on computer graphics with an interested student.

Chapter 2 SUPPLEMENTARY EXERCISES

1. a. True. If A and B are $m\times n$ matrices, then B^T has as many rows as A has columns, so AB^T is defined. Also, A^TB is defined because A^T has m columns and B has m rows.

b. False. B must have 2 columns. A has as many columns as B has rows.

c. True. The ith row of A has the form $(0, \ldots, d_i, \ldots, 0)$. So the ith row of AB is $(0, \ldots, d_i, \ldots, 0)B$, which is d_i times the ith row of B.

d. False. Take the zero matrix for B. Or, construct a matrix B such that the equation $B\mathbf{x} = \mathbf{0}$ has nontrivial solutions, and construct C and D so that $C \neq D$ and the columns of $C - D$ satisfy the equation $B\mathbf{x} = \mathbf{0}$. Then $B(C - D) = 0$ and $BC = BD$.

e. False. Counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

f. False. $(A + B)(A - B) = A^2 - AB + BA - B^2$. This equals $A^2 - B^2$ if and only if A commutes with B.

g. True. An $n\times n$ replacement matrix has $n + 1$ nonzero entries. The $n\times n$ scale and interchange matrices have n nonzero entries.

h. True. The transpose of an elementary matrix is an elementary matrix of the same type.

i. True. An $n\times n$ elementary matrix is obtained by a row operation on I_n.

j. False. Elementary matrices are invertible, so a product of such matrices is invertible. But not every square matrix is invertible.

k. True. If A is 3×3 with three pivot positions, then A is row equivalent to I_3.

l. False. A must be square in order to conclude from the equation $AB = I$ that A is invertible.

m. False. AB is invertible, but $(AB)^{-1} = B^{-1}A^{-1}$, and this product is not always equal to $A^{-1}B^{-1}$.

n. True. Given $AB = BA$, left-multiply by A^{-1} to get $B = A^{-1}BA$, and then right-multiply by A^{-1} to obtain $BA^{-1} = A^{-1}B$.

o. False. The correct equation is $(rA)^{-1} = r^{-1}A^{-1}$, because

$(rA)(r^{-1}A^{-1}) = (rr^{-1})(AA^{-1}) = 1 \cdot I = I$.

p. True. If the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has a unique solution, then there are no free variables in this equation, which means that A must have three pivot positions (since A is 3×3). By the Invertible Matrix Theorem, A is invertible.

2. $C = (C^{-1})^{-1} = \dfrac{1}{-2}\begin{bmatrix} 7 & -5 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} -7/2 & 5/2 \\ 3 & -2 \end{bmatrix}$

3. $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$A^3 = A \cdot A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Next, $(I - A)(I + A + A^2) = I + A + A^2 - A(I + A + A^2) = I + A + A^2 - A - A^2 - A^3 = I - A^3$.

Since $A^3 = 0$, $(I - A)(I + A + A^2) = I$.

4. From Exercise 3, the inverse of $I - A$ is probably $I + A + A^2 + \cdots + A^{n-1}$. To verify this, compute

$$(I - A)(I + A + \cdots + A^{n-1}) = I + A + \cdots + A^{n-1} - A(I + A + \cdots + A^{n-1}) = I - AA^{n-1} = I - A^n$$

If $A^n = 0$, then the matrix $B = I + A + A^2 + \cdots + A^{n-1}$ satisfies $(I - A)B = I$. Since $I - A$ and B are square, they are invertible by the Invertible Matrix Theorem, and B is the inverse of $I - A$.

5. $A^2 = 2A - I$. Multiply by A: $A^3 = 2A^2 - A$. Substitute $A^2 = 2A - I$: $A^3 = 2(2A - I) - A = 3A - 2I$. Multiply by A again: $A^4 = A(3A - 2I) = 3A^2 - 2A$. Substitute the identity $A^2 = 2A - I$ again: Finally, $A^4 = 3(2A - I) - 2A = 4A - 3I$.

6. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. By direct computation, $A^2 = I$, $B^2 = I$, and $AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -BA$.

7. (Partial answer in *Study Guide*) Since $A^{-1}B$ is the solution of $AX = B$, row reduction of $[A \;\; B]$ to $[I \;\; X]$ will produce $X = A^{-1}B$. See Exercise 15 in Section 2.2.

$$[A \;\; B] = \left[\begin{array}{ccc|cc} 1 & 3 & 8 & -3 & 5 \\ 2 & 4 & 11 & 1 & 5 \\ 1 & 2 & 5 & 3 & 4 \end{array}\right] \sim \begin{bmatrix} 1 & 3 & 8 & -3 & 5 \\ 0 & -2 & -5 & 7 & -5 \\ 0 & -1 & -3 & 6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 8 & -3 & 5 \\ 0 & 1 & 3 & -6 & 1 \\ 0 & -2 & -5 & 7 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 8 & -3 & 5 \\ 0 & 1 & 3 & -6 & 1 \\ 0 & 0 & 1 & -5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 37 & 29 \\ 0 & 1 & 0 & 9 & 10 \\ 0 & 0 & 1 & -5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 10 & -1 \\ 0 & 1 & 0 & 9 & 10 \\ 0 & 0 & 1 & -5 & -3 \end{bmatrix}$$

Thus, $A^{-1}B = \begin{bmatrix} 10 & -1 \\ 9 & 10 \\ -5 & -3 \end{bmatrix}$.

8. By definition of matrix multiplication, the matrix A satisfies

$$A\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$$

Right-multiply both sides by the inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$. The left side becomes A. Thus,

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 4 & -1 \end{bmatrix}$$

9. Given $AB = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$, notice that $ABB^{-1} = A$. Since $\det B = 7 - 6 = 1$,

$$B^{-1} = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} \text{ and } A = (AB)B^{-1} = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} -3 & 13 \\ -8 & 27 \end{bmatrix}$$

Note: Variants of this question make simple exam questions.

10. Since A is invertible, so is A^T, by the Invertible Matrix Theorem. Then A^TA is the product of invertible matrices and so is invertible. Thus, the formula $(A^TA)^{-1}A^T$ makes sense. By Theorem 6 in Section 2.2,

$$(A^TA)^{-1}\cdot A^T = A^{-1}(A^T)^{-1}A^T = A^{-1}I = A^{-1}$$

An alternative calculation: $(A^TA)^{-1}A^T\cdot A = (A^TA)^{-1}(A^TA) = I$. Since A is invertible, this equation shows that its inverse is $(A^TA)^{-1}A^T$.

11. **a.** For $i = 1,\dots, n$, $p(x_i) = c_0 + c_1x_i + \cdots + c_{n-1}x_i^{n-1} = \text{row}_i(V)\cdot\begin{bmatrix} c_0 \\ \vdots \\ c_{n-1}\end{bmatrix} = \text{row}_i(V)\mathbf{c}$.

By a property of matrix multiplication, shown after Example 6 in Section 2.1, and the fact that **c** was chosen to satisfy $V\mathbf{c} = \mathbf{y}$,

$$\text{row}_i(V)\mathbf{c} = \text{row}_i(V\mathbf{c}) = \text{row}_i(\mathbf{y}) = y_i$$

Thus, $p(x_i) = y_i$. To summarize, the entries in $V\mathbf{c}$ are the values of the polynomial $p(x)$ at $x_1, \dots, x_n$.

b. Suppose $x_1, \dots, x_n$ are distinct, and suppose $V\mathbf{c} = \mathbf{0}$ for some vector **c**. Then the entries in **c** are the coefficients of a polynomial whose value is zero at the distinct points $x_1, \dots, x_n$. However, a nonzero polynomial of degree $n - 1$ cannot have n zeros, so the polynomial must be identically zero. That is, the entries in **c** must all be zero. This shows that the columns of V are linearly independent.

c. (Solution in *Study Guide*) When $x_1, \dots, x_n$ are distinct, the columns of V are linearly independent, by (b). By the Invertible Matrix Theorem, V is invertible and its columns span $\mathbf{R}^n$. So, for every $\mathbf{y} = (y_1, \dots, y_n)$ in $\mathbf{R}^n$, there is a vector **c** such that $V\mathbf{c} = \mathbf{y}$. Let p be the polynomial whose coefficients are listed in **c**. Then, by (a), p is an interpolating polynomial for $(x_1, y_1), \dots, (x_n, y_n)$.

12. If $A = LU$, then $\text{col}_1(A) = L\cdot\text{col}_1(U)$. Since $\text{col}_1(U)$ has a zero in every entry except possibly the first, $L\cdot\text{col}_1(U)$ is a linear combination of the columns of L in which all weights except possibly the first are zero. So $\text{col}_1(A)$ is a multiple of $\text{col}_1(L)$.

Similarly, $\text{col}_2(A) = L\cdot\text{col}_2(U)$, which is a linear combination of the columns of L using the first two entries in $\text{col}_2(U)$ as weights, because the other entries in $\text{col}_2(U)$ are zero. Thus $\text{col}_2(A)$ is a linear combination of the first two columns of L.

13. **a.** $P^2 = (\mathbf{u}\mathbf{u}^T)(\mathbf{u}\mathbf{u}^T) = \mathbf{u}(\mathbf{u}^T\mathbf{u})\mathbf{u}^T = \mathbf{u}(1)\mathbf{u}^T = P$, because **u** satisfies $\mathbf{u}^T\mathbf{u} = 1$.

b. $P^T = (\mathbf{u}\mathbf{u}^T)^T = \mathbf{u}^{TT}\mathbf{u}^T = \mathbf{u}\mathbf{u}^T = P$

c. $Q^2 = (I - 2P)(I - 2P) = I - I(2P) - 2PI + 2P(2P)$
$= I - 4P + 4P^2 = I$, because of part (a).

14. Given $\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 1\end{bmatrix}$, define P and Q as in Exercise 13 by

$$P = \mathbf{u}\mathbf{u}^T = \begin{bmatrix} 0 \\ 0 \\ 1\end{bmatrix}\begin{bmatrix} 0 & 0 & 1\end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{bmatrix},\quad Q = I - 2P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix} - 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1\end{bmatrix}$$

If $\mathbf{x} = \begin{bmatrix} 1 \\ 5 \\ 3\end{bmatrix}$, then $P\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} 1 \\ 5 \\ 3\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3\end{bmatrix}$ and $Q\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix} 1 \\ 5 \\ 3\end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -3\end{bmatrix}$.

15. Left-multiplication by an elementary matrix produces an elementary row operation:

$$B \sim E_1B \sim E_2E_1B \sim E_3E_2E_1B = C$$

so B is row equivalent to C. Since row operations are reversible, C is row equivalent to B. (Alternatively, show C being changed into B by row operations using the inverse of the E_i .)

16. Since A is not invertible, there is a nonzero vector **v** in $\mathbf{R}^n$ such that $A\mathbf{v} = \mathbf{0}$. Place n copies of **v** into an $n\times n$ matrix B. Then $AB = A[\mathbf{v} \cdots \mathbf{v}] = [A\mathbf{v} \cdots A\mathbf{v}] = 0$.

17. Let A be a 6×4 matrix and B a 4×6 matrix. Since B has more columns than rows, its six columns are linearly dependent and there is a nonzero **x** such that $B\mathbf{x} = \mathbf{0}$. Thus $AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$. This shows that the matrix AB is not invertible, by the IMT. (Basically the same argument was used to solve Exercise 22 in Section 2.1.)

Note: (In the *Study Guide*) It is possible that BA is invertible. For example, let C be an invertible 4×4 matrix and construct $A = \begin{bmatrix} C \\ 0\end{bmatrix}$ and $B = [C^{-1}\ \ 0]$. Then $BA = I_4$, which is invertible.

18. By hypothesis, A is 5×3, C is 3×5, and $CA = I_3$. Suppose **x** satisfies $A\mathbf{x} = \mathbf{b}$. Then $CA\mathbf{x} = C\mathbf{b}$. Since $CA = I$, **x** must be $C\mathbf{b}$. This shows that $C\mathbf{b}$ is the only solution of $A\mathbf{x} = \mathbf{b}$.

19. **[M]** Let $A = \begin{bmatrix} .4 & .2 & .3 \\ .3 & .6 & .3 \\ .3 & .2 & .4\end{bmatrix}$. Then $A^2 = \begin{bmatrix} .31 & .26 & .30 \\ .39 & .48 & .39 \\ .30 & .26 & .31\end{bmatrix}$. Instead of computing A^3 next, speed up the calculations by computing

$$A^4 = A^2A^2 = \begin{bmatrix} .2875 & .2834 & .2874 \\ .4251 & .4332 & .4251 \\ .2874 & .2834 & .2875\end{bmatrix},\quad A^8 = A^4A^4 = \begin{bmatrix} .2857 & .2857 & .2857 \\ .4285 & .4286 & .4285 \\ .2857 & .2857 & .2857\end{bmatrix}$$

To four decimal places, as k increases,

$$A^k \to \begin{bmatrix} .2857 & .2857 & .2857 \\ .4286 & .4286 & .4286 \\ .2857 & .2857 & .2857\end{bmatrix},\ \text{or, in rational format,}\ A^k \to \begin{bmatrix} 2/7 & 2/7 & 2/7 \\ 3/7 & 3/7 & 3/7 \\ 2/7 & 2/7 & 2/7\end{bmatrix}.$$

If $B = \begin{bmatrix} 0 & .2 & .3 \\ .1 & .6 & .3 \\ .9 & .2 & .4\end{bmatrix}$, then $B^2 = \begin{bmatrix} .29 & .18 & .18 \\ .33 & .44 & .33 \\ .38 & .38 & .49\end{bmatrix}$,

$$B^4 = \begin{bmatrix} .2119 & .1998 & .1998 \\ .3663 & .3784 & .3663 \\ .4218 & .4218 & .4339 \end{bmatrix}, \quad B^8 = \begin{bmatrix} .2024 & .2022 & .2022 \\ .3707 & .3709 & .3707 \\ .4269 & .4269 & .4271 \end{bmatrix}$$

To four decimal places, as k increases,

$$B^k \to \begin{bmatrix} .2022 & .2022 & .2022 \\ .3708 & .3708 & .3708 \\ .4270 & .4270 & .4270 \end{bmatrix}, \text{ or, in rational format, } B^k \to \begin{bmatrix} 18/89 & 18/89 & 18/89 \\ 33/89 & 33/89 & 33/89 \\ 38/89 & 38/89 & 38/89 \end{bmatrix}.$$

20. **[M]** The 4×4 matrix A_4 is the 4×4 matrix of ones, minus the 4×4 identity matrix. The MATLAB command is `A4 = ones(4) - eye(4)`. For the inverse, use `inv(A4)`.

$$A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad A_4^{-1} = \begin{bmatrix} -2/3 & 1/3 & 1/3 & 1/3 \\ 1/3 & -2/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & 1/3 & -2/3 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad A_5^{-1} = \begin{bmatrix} -3/4 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & -3/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & -3/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & -3/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 & -3/4 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad A_6^{-1} = \begin{bmatrix} -4/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & -4/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & -4/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & -4/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & -4/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & -4/5 \end{bmatrix}$$

The construction of A_6 and the appearance of its inverse suggest that the inverse is related to I_6. In fact, $A_6^{-1} + I_6$ is 1/5 times the 6×6 matrix of ones. Let J denotes the $n \times n$ matrix of ones. The conjecture is:

$$A_n = J - I_n \quad \text{and} \quad A_n^{-1} = \frac{1}{n-1} \cdot J - I_n$$

Proof: (Not required) Observe that $J^2 = nJ$ and $A_n J = (J - I)J = J^2 - J = (n - 1) J$. Now compute

$$A_n((n-1)^{-1}J - I) = (n-1)^{-1}A_n J - A_n = J - (J - I) = I$$

Since A_n is square, A_n is invertible and its inverse is $(n-1)^{-1}J - I$.