$$D^{-1} = \frac{100}{3}\begin{bmatrix} 3 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

To find the forces (in pounds) required to produce a deflection of .04 cm at point 3, most students will use technology to solve $D\mathbf{f} = (0, 0, .04)$ and obtain $(0, -4/3, 4)$.

Here is another method, based on the idea suggested in Exercise 42. The first column of $D^{-1}$ lists the forces required to produce a deflection of 1 in. at point 1 (with zero deflection at the other points). Since the transformation $\mathbf{y} \mapsto D^{-1}\mathbf{y}$ is linear, the forces required to produce a deflection of .04 cm at point 3 is given by .04 times the third column of $D^{-1}$, namely (.04)(100/3) times $(0, -1, 3)$, or $(0, -4/3, 4)$ pounds.

**41.** To determine the forces that produce deflections of .07, .12, .16, and .12 cm at the four points on the beam, use technology to solve $D\mathbf{f} = \mathbf{y}$, where $\mathbf{y} = (.07, .12, .16, .12)$. The forces at the four points are .95, 6.19, 11.43, and 3.81 newtons, respectively.

**42.** **[M]** To determine the forces that produce a deflection of .22 cm at the second point on the beam, use technology to solve $D\mathbf{f} = \mathbf{y}$, where $\mathbf{y} = (0, .22, 0, 0)$. The forces at the four points are –10.476, 31.429,
–10.476, and 0 newtons, respectively (to three significant digits). These forces are .22 times the entries in the second column of $D^{-1}$. *Reason*: The transformation $\mathbf{y} \mapsto D^{-1}\mathbf{y}$ is linear, so the forces required to produce a deflection of .22 cm at the second point are .22 times the forces required to produce a deflection of 1 cm at the second point. These forces are listed in the second column of $D^{-1}$.
Another possible discussion: The solution of $D\mathbf{x} = (0, 1, 0, 0)$ is the second column of $D^{-1}$. Multiply both sides of this equation by .22 to obtain $D(.22\mathbf{x}) = (0, .22, 0, 0)$. So $.22\mathbf{x}$ is the solution of $D\mathbf{f} = (0, .22, 0, 0)$. (The argument uses linearity, but students may not mention this.)

**Note**: The *Study Guide* suggests using **`gauss`**, **`swap`**, **`bgauss`**, and **`scale`** to reduce $[A \quad I]$ because I prefer to postpone the use of **`ref`** (or **`rref`**) until later. If you wish to introduce **`ref`** now, see the *Study Guide*'s technology notes for Sections 2.8 or 4.3. (Recall that Sections 2.8 and 2.9 are only covered when an instructor plans to skip Chapter 4 and get quickly to eigenvalues.)

## 2.3 SOLUTIONS

**Notes**: This section ties together most of the concepts studied thus far. With strong encouragement from an instructor, most students can use this opportunity to review and reflect upon what they have learned, and form a solid foundation for future work. Students who fail to do this now usually struggle throughout the rest of the course. Section 2.3 can be used in at least three different ways.

(1) Stop after Example 1 and assign exercises only from among the Practice Problems and Exercises 1 to 28. I do this when teaching "Course 3" described in the text's "Notes to the Instructor. " If you did not cover Theorem 12 in Section 1.9, omit statements (f) and (i) from the Invertible Matrix Theorem.

(2) Include the subsection "Invertible Linear Transformations" in Section 2.3, if you covered Section 1.9. I do this when teaching "Course 1" because our mathematics and computer science majors take this class. Exercises 29–40 support this material.

(3) Skip the linear transformation material here, but discusses the **condition number** and the Numerical Notes. Assign exercises from among 1–28 and 41–45, and perhaps add a computer project on the condition number. (See the projects on our web site.) I do this when teaching "Course 2" for our engineers.

The abbreviation IMT (here and in the *Study Guide*) denotes the Invertible Matrix Theorem (Theorem 8).

**1.** The columns of the matrix $\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$ are not multiples, so they are linearly independent. By (e) in the IMT, the matrix is invertible. Also, the matrix is invertible by Theorem 4 in Section 2.2 because the determinant is nonzero.

**2.** The fact that the columns of $\begin{bmatrix} -4 & 2 \\ 6 & -3 \end{bmatrix}$ are multiples of each other is one way to show that this matrix is not invertible. Another is to check the determinant. In this case it is easily seen to be zero. By Theorem 4 in Section 2.2, the matrix is not invertible.

**3.** Row reduction to echelon form is trivial because there is really no need for arithmetic calculations:
$\begin{bmatrix} 3 & 0 & 0 \\ -3 & -4 & 0 \\ 8 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ The 3×3 matrix has 3 pivot positions and hence is invertible, by (c) of the IMT. [Another explanation could be given using the transposed matrix. But see the note below that follows the solution of Exercise 14.]

**4.** The matrix $\begin{bmatrix} -5 & 1 & 4 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix}$ cannot row reduce to the identity matrix since it already contains a row of zeros. Hence the matrix is not invertible (or singular) by (b) in the IMT.

**5.** The matrix $\begin{bmatrix} 3 & 0 & -3 \\ 2 & 0 & 4 \\ -4 & 0 & 7 \end{bmatrix}$ obviously has linearly dependent columns (because one column is zero), and so the matrix is not invertible (or singular) by (e) in the IMT.

**6.** $\begin{bmatrix} 1 & -3 & -6 \\ 0 & 4 & 3 \\ -3 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -6 \\ 0 & 4 & 3 \\ 0 & -3 & -18 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -6 \\ 0 & 4 & 3 \\ 0 & 1 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -6 \\ 0 & 4 & 3 \\ 0 & 0 & -21 \end{bmatrix}$

The matrix is invertible because it is row equivalent to the identity matrix.

**7.** $\begin{bmatrix} -1 & -3 & 0 & 1 \\ 3 & 5 & 8 & -3 \\ -2 & -6 & 3 & 2 \\ 0 & -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & -3 & 0 & 1 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & -3 & 0 & 1 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The 4×4 matrix has four pivot positions and so is invertible by (c) of the IMT.

8. The 4×4 matrix $\begin{bmatrix} 3 & 4 & 7 & 4 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is invertible because it has four pivot positions, by (c) of the IMT.

9. **[M]** Using technology, $\begin{bmatrix} 4 & 0 & -3 & -7 \\ -6 & 9 & 9 & 9 \\ 7 & -5 & 10 & 19 \\ -1 & 2 & 4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

The 4×4 matrix is invertible because it has four pivot positions, by (c) of the IMT.

10. **[M]** $\begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 6 & 4 & 2 & 8 & -8 \\ 7 & 5 & 3 & 10 & 9 \\ 9 & 6 & 4 & -9 & -5 \\ 8 & 5 & 2 & 11 & 4 \end{bmatrix} \sim \begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 0 & .4 & .8 & -.4 & -18.8 \\ 0 & .8 & 1.6 & .2 & -3.6 \\ 0 & .6 & 2.2 & -21.6 & -21.2 \\ 0 & .2 & .4 & -.2 & -10.4 \end{bmatrix}$

$$\sim \begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 0 & .4 & .8 & -.4 & -18.8 \\ 0 & 0 & 0 & 1 & 34 \\ 0 & 0 & 1 & -21 & 7 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 0 & .4 & .8 & -.4 & -18.8 \\ 0 & 0 & 1 & -21 & 7 \\ 0 & 0 & 0 & 1 & 34 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

The 5×5 matrix is invertible because it has five pivot positions, by (c) of the IMT.

11. **a.** True, by the IMT. If statement (d) of the IMT is true, then so is statement (b).
    **b.** True. If statement (h) of the IMT is true, then so is statement (e).
    **c.** False. Statement (g) of the IMT is true only for invertible matrices.
    **d.** True, by the IMT. If the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then statement (d) of the IMT is false. In this case, all the lettered statements in the IMT are false, including statement (c), which means that $A$ must have fewer than $n$ pivot positions.
    **e.** True, by the IMT. If $A^T$ is not invertible, then statement (1) of the IMT is false, and hence statement (a) must also be false.

12. **a.** True. If statement (k) of the IMT is true, then so is statement ( j). Use the first box after the IMT.
    **b.** False. Notice that (i) if the IMT uses the work *onto* rather than the word *into*.
    **c.** True. If statement (e) of the IMT is true, then so is statement (h).
    **d.** False. Since (g) if the IMT is true, so is (f).
    **e.** False, by the IMT. The fact that there is a $\mathbf{b}$ in $\mathbb{R}^n$ such that the equation $A\mathbf{x} = \mathbf{b}$ is consistent, does not imply that statement (g) of the IMT is true, and hence there could be more than one solution.

**Note:** The solutions below for Exercises 13–30 refer mostly to the IMT. In many cases, however, part or all of an acceptable solution could also be based on various results that were used to establish the IMT.

13. If a square upper triangular $n \times n$ matrix has nonzero diagonal entries, then because it is already in echelon form, the matrix is row equivalent to $I_n$ and hence is invertible, by the IMT. Conversely, if the matrix is invertible, it has $n$ pivots on the diagonal and hence the diagonal entries are nonzero.

14. If $A$ is lower triangular with nonzero entries on the diagonal, then these $n$ diagonal entries can be used as pivots to produce zeros below the diagonal. Thus $A$ has $n$ pivots and so is invertible, by the IMT. If one of the diagonal entries in $A$ is zero, $A$ will have fewer than $n$ pivots and hence be singular.

**Notes:** For Exercise 14, another correct analysis of the case when $A$ has nonzero diagonal entries is to apply the IMT (or Exercise 13) to $A^T$. Then use Theorem 6 in Section 2.2 to conclude that since $A^T$ is invertible so is its transpose, $A$. You might mention this idea in class, but I recommend that you not spend much time discussing $A^T$ and problems related to it, in order to keep from making this section too lengthy. (The transpose is treated infrequently in the text until Chapter 6.)

If you do plan to ask a test question that involves $A^T$ and the IMT, then you should give the students some extra homework that develops skill using $A^T$. For instance, in Exercise 14 replace "columns" by "rows." Also, you could ask students to explain why an $n \times n$ matrix with linearly independent columns must also have linearly independent rows.

15. Part (h) of the IMT shows that a 4×4 matrix cannot be invertible when its columns do not span $\mathbf{R}^4$.

16. If $A$ is invertible, so is $A^T$, by (l) of the IMT. By (e) of the IMT applied to $A^T$, the columns of $A^T$ are linearly independent.

17. If $A$ has two identical columns then its columns are linearly dependent. Part (e) of the IMT shows that $A$ cannot be invertible.

18. If $A$ contains two identical rows, then it cannot be row reduced to the identity because subtracting one row from the other creates a row of zeros. By (b) of the IMT, such a matrix cannot be invertible.

19. By (e) of the IMT, $D$ is invertible. Thus the equation $D\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^7$, by (g) of the IMT. Even better, the equation $D\mathbf{x} = \mathbf{b}$ has a *unique* solution for each $\mathbf{b}$ in $\mathbf{R}^7$, by Theorem 5 in Section 2.2. (See the paragraph following the proof of the IMT.)

20. By (g) of the IMT, $A$ is invertible. Hence, each equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, by Theorem 5 in Section 2.2. This fact was pointed out in the paragraph following the proof of the IMT.

21. The matrix $C$ cannot be invertible, by Theorem 5 in Section 2.2 or by the box following the IMT. So (h) of the IMT is false and the columns of $C$ do not span $\mathbf{R}^n$.

22. By the box following the IMT, $E$ and $F$ are invertible and are inverses. So $FE = I = EF$, and so $E$ and $F$ commute.

23. Statement (g) of the IMT is false for $F$, so statement (d) is false, too. That is, the equation $F\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

24. Statement (b) of the IMT is false for G, so statements (e) and (h) are also false. That is, the columns of G are linearly *de*pendent and the columns do *not* span $\mathbf{R}^n$.

25. Suppose that $A$ is square and $AB = I$. Then $A$ is invertible, by the (k) of the IMT. Left-multiplying each side of the equation $AB = I$ by $A^{-1}$, one has

$$A^{-1}AB = A^{-1}I, \quad IB = A^{-1}, \quad \text{and } B = A^{-1}.$$

By Theorem 6 in Section 2.2, the matrix $B$ (which is $A^{-1}$) is invertible, and its inverse is $(A^{-1})^{-1}$, which is $A$.

**26.** If the columns of $A$ are linearly independent, then since $A$ is square, $A$ is invertible, by the IMT. So $A^2$, which is the product of invertible matrices, is invertible. By the IMT, the columns of $A^2$ span $\mathbf{R}^n$.

**27.** Let $W$ be the inverse of $AB$. Then $ABW = I$ and $A(BW) = I$. Since $A$ is square, $A$ is invertible, by (k) of the IMT.

**Note**: The *Study Guide* for Exercise 27 emphasizes here that the equation $A(BW) = I$, *by itself*, does not show that $A$ is invertible. Students are referred to Exercise 38 in Section 2.2 for a counterexample. Although there is an overall assumption that matrices in this section are square, I insist that my students mention this fact when using the IMT. Even so, at the end of the course, I still sometimes find a student who thinks that an equation $AB = I$ implies that $A$ is invertible.

**28.** Let $W$ be the inverse of $AB$. Then $WAB = I$ and $(WA)B = I$. By (j) of the IMT applied to $B$ in place of $A$, the matrix $B$ is invertible.

**29.** Since the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one, statement (f) of the IMT is true. Then (i) is also true and the transformation $\mathbf{x} \mapsto A\mathbf{x}$ does map $\mathbf{R}^n$ *onto* $\mathbf{R}^n$. Also, $A$ is invertible, which implies that the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is invertible, by Theorem 9.

**30.** Since the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not one-to-one, statement (f) of the IMT is false. Then (i) is also false and the transformation $\mathbf{x} \mapsto A\mathbf{x}$ does not map $\mathbf{R}^n$ onto $\mathbf{R}^n$. Also, $A$ is not invertible, which implies that the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is not invertible, by Theorem 9.

**31.** Since the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$, the matrix $A$ has a pivot in each row (Theorem 4 in Section 1.4). Since $A$ is square, $A$ has a pivot in each column, and so there are no free variables in the equation $A\mathbf{x} = \mathbf{b}$, which shows that the solution is unique.

**Note**: The preceding argument shows that the (square) shape of $A$ plays a crucial role. A less revealing proof is to use the "pivot in each row" and the IMT to conclude that $A$ is invertible. Then Theorem 5 in Section 2.2 shows that the solution of $A\mathbf{x} = \mathbf{b}$ is unique.

**32.** If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then $A$ must have a pivot in each of its $n$ columns. Since $A$ is square (and this is the key point), there must be a pivot in each *row* of $A$. By Theorem 4 in Section 1.4, the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^n$.

Another argument: Statement (d) of the IMT is true, so $A$ is invertible. By Theorem 5 in Section 2.2, the equation $A\mathbf{x} = \mathbf{b}$ has a (unique) solution for each $\mathbf{b}$ in $\mathbf{R}^n$.

**33.** (Solution in *Study Guide*) The standard matrix of $T$ is $A = \begin{bmatrix} -5 & 9 \\ 4 & -7 \end{bmatrix}$, which is invertible because $\det A \neq 0$. By Theorem 9, the transformation $T$ is invertible and the standard matrix of $T^{-1}$ is $A^{-1}$. From the formula for a 2×2 inverse, $A^{-1} = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}$. So

$$T^{-1}(x_1,x_2) = \begin{bmatrix} 7 & 9 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (7x_1 + 9x_2, 4x_1 + 5x_2)$$

**34.** The standard matrix of $T$ is $A = \begin{bmatrix} 2 & -8 \\ -2 & 7 \end{bmatrix}$, which is invertible because $\det A = -2 \neq 0$. By Theorem 9, $T$ is invertible, and $T^{-1}(\mathbf{x}) = B\mathbf{x}$, where $B = A^{-1} = -\frac{1}{2}\begin{bmatrix} 7 & 8 \\ 2 & 2 \end{bmatrix}$. Thus

$$T^{-1}(x_1,x_2) = -\frac{1}{2}\begin{bmatrix} 7 & 8 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left(-\frac{7}{2}x_1 - 4x_2, -x_1 - x_2\right)$$

**35.** (Solution in *Study Guide*) To show that $T$ is one-to-one, suppose that $T(\mathbf{u}) = T(\mathbf{v})$ for some vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^n$. Then $S(T(\mathbf{u})) = S(T(\mathbf{v}))$, where $S$ is the inverse of $T$. By Equation (1), $\mathbf{u} = S(T(\mathbf{u}))$ and $S(T(\mathbf{v})) = \mathbf{v}$, so $\mathbf{u} = \mathbf{v}$. Thus $T$ is one-to-one. To show that $T$ is onto, suppose $\mathbf{y}$ represents an arbitrary vector in $\mathbf{R}^n$ and define $\mathbf{x} = S(\mathbf{y})$. Then, using Equation (2), $T(\mathbf{x}) = T(S(\mathbf{y})) = \mathbf{y}$, which shows that $T$ maps $\mathbf{R}^n$ onto $\mathbf{R}^n$.

Second proof: By Theorem 9, the standard matrix $A$ of $T$ is invertible. By the IMT, the columns of $A$ are linearly independent and span $\mathbf{R}^n$. By Theorem 12 in Section 1.9, $T$ is one-to-one and maps $\mathbf{R}^n$ onto $\mathbf{R}^n$.

**36.** Let $A$ be the standard matrix of $T$. By hypothesis, $T$ is not a one-to-one mapping. So, by Theorem 12 in Section 1.9, the standard matrix $A$ of $T$ has linearly dependent columns. Since $A$ is square, the columns of $A$ do not span $\mathbf{R}^n$. By Theorem 12, again, $T$ cannot map $\mathbf{R}^n$ onto $\mathbf{R}^n$.

**37.** Let $A$ and $B$ be the standard matrices of $T$ and $U$, respectively. Then $AB$ is the standard matrix of the mapping $\mathbf{x} \mapsto T(U(\mathbf{x}))$, because of the way matrix multiplication is defined (in Section 2.1). By hypothesis, this mapping is the identity mapping, so $AB = I$. Since $A$ and $B$ are square, they are invertible, by the IMT, and $B = A^{-1}$. Thus, $BA = I$. This means that the mapping $\mathbf{x} \mapsto U(T(\mathbf{x}))$ is the identity mapping, i.e., $U(T(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbf{R}^n$.

**38.** Given any $\mathbf{v}$ in $\mathbf{R}^n$, we may write $\mathbf{v} = T(\mathbf{x})$ for some $\mathbf{x}$, because $T$ is an onto mapping. Then, the assumed properties of $S$ and $U$ show that $S(\mathbf{v}) = S(T(\mathbf{x})) = \mathbf{x}$ and $U(\mathbf{v}) = U(T(\mathbf{x})) = \mathbf{x}$. So $S(\mathbf{v})$ and $U(\mathbf{v})$ are equal for each $\mathbf{v}$. That is, $S$ and $U$ are the same function from $\mathbf{R}^n$ into $\mathbf{R}^n$.

**39.** If $T$ maps $\mathbf{R}^n$ onto $\mathbf{R}^n$, then the columns of its standard matrix $A$ span $\mathbf{R}^n$, by Theorem 12 in Section 1.9. By the IMT, $A$ is invertible. Hence, by Theorem 9 in Section 2.3, $T$ is invertible, and $A^{-1}$ is the standard matrix of $T^{-1}$. Since $A^{-1}$ is also invertible, by the IMT, its columns are linearly independent and span $\mathbf{R}^n$. Applying Theorem 12 in Section 1.9 to the transformation $T^{-1}$, we conclude that $T^{-1}$ is a one-to-one mapping of $\mathbf{R}^n$ onto $\mathbf{R}^n$.

**40.** Given $\mathbf{u}$, $\mathbf{v}$ in $\mathbf{R}^n$, let $\mathbf{x} = S(\mathbf{u})$ and $\mathbf{y} = S(\mathbf{v})$. Then $T(\mathbf{x})=T(S(\mathbf{u})) = \mathbf{u}$ and $T(\mathbf{y}) = T(S(\mathbf{v})) = \mathbf{v}$, by equation (2). Hence

$$\begin{aligned} S(\mathbf{u}+\mathbf{v}) &= S(T(\mathbf{x})+T(\mathbf{y})) & \\ &= S(T(\mathbf{x}+\mathbf{y})) & \text{Because } T \text{ is linear} \\ &= \mathbf{x}+\mathbf{y} & \text{By equation (1)} \\ &= S(\mathbf{u})+S(\mathbf{v}) & \end{aligned}$$

So, $S$ preserves sums. For any scalar $r$,

$$\begin{aligned} S(r\mathbf{u}) &= S(rT(\mathbf{x})) = S(T(r\mathbf{x})) && \text{Because } T \text{ is linear} \\ &= r\mathbf{x} && \text{By equation (1)} \\ &= rS(\mathbf{u}) \end{aligned}$$

So $S$ preserves scalar multiples. Thus $S$ is a linear transformation.

**41.** **[M]** **a.** The exact solution of (3) is $x_1 = 3.94$ and $x_2 = .49$. The exact solution of (4) is $x_1 = 2.90$ and $x_2 = 2.00$.

**b.** When the solution of (4) is used as an approximation for the solution in (3), the error in using the value of 2.90 for $x_1$ is about 26%, and the error in using 2.0 for $x_2$ is about 308%.

**c.** The condition number of the coefficient matrix is 3363. The percentage change in the solution from (3) to (4) is about 7700 times the percentage change in the right side of the equation. This is the same order of magnitude as the condition number. The condition number gives a rough measure of how sensitive the solution of $A\mathbf{x} = \mathbf{b}$ can be to changes in $\mathbf{b}$. Further information about the condition number is given at the end of Chapter 6 and in Chapter 7.

**Note**: See the *Study Guide*'s MATLAB box, or a technology appendix, for information on condition number. Only the TI-83+ and TI-89 lack a command for this.

**42.** **[M]** MATLAB gives $\text{cond}(A) \approx 10$, which is approximately $10^1$. If you make several trials with MATLAB, which records 16 digits accurately, you should find that $\mathbf{x}$ and $\mathbf{x}_1$ agree to at least 14 or 15 significant digits. So about 1 significant digit is lost. Here is the result of one experiment. The vectors were all computed to the maximum 16 decimal places but are here displayed with only four decimal places:

$$\mathbf{x} = \text{rand}(4,1) = \begin{bmatrix} .9501 \\ .2311 \\ .6068 \\ .4860 \end{bmatrix}, \mathbf{b} = A\mathbf{x} = \begin{bmatrix} -1.4219 \\ 6.2149 \\ 20.7973 \\ 1.4535 \end{bmatrix}. \text{ The MATLAB solution is } \mathbf{x}_1 = A\backslash\mathbf{b} = \begin{bmatrix} .9501 \\ .2311 \\ .6068 \\ .4860 \end{bmatrix}.$$

However, $\mathbf{x} - \mathbf{x}_1 = \begin{bmatrix} -.2220 \\ -.2220 \\ 0 \\ -.1665 \end{bmatrix} \times 10^{-15}$. The computed solution $\mathbf{x}_1$ is accurate to about 14 decimal places.

**43.** **[M]** MATLAB gives $\text{cond}(A) = 69{,}000$. Since this has magnitude between $10^4$ and $10^5$, the estimated accuracy of a solution of $A\mathbf{x} = \mathbf{b}$ should be to about four or five decimal places *less* than the 16 decimal places that MATLAB usually computes accurately. That is, one should expect the solution to be accurate to only about 11 or 12 decimal places. Here is the result of one experiment. The vectors were all computed to the maximum 16 decimal places but are here displayed with only four decimal places:

$$\mathbf{x} = \text{rand}(5,1) = \begin{bmatrix} .8214 \\ .4447 \\ .6154 \\ .7919 \\ .9218 \end{bmatrix}, \mathbf{b} = A\mathbf{x} = \begin{bmatrix} 19.8965 \\ 6.8991 \\ 26.0354 \\ 0.7861 \\ 22.4242 \end{bmatrix}. \text{ The MATLAB solution is } \mathbf{x}_1 = A\backslash\mathbf{b} = \begin{bmatrix} .8214 \\ .4447 \\ .6154 \\ .7919 \\ .9218 \end{bmatrix}.$$

However, $\mathbf{x} - \mathbf{x}_1 = \begin{bmatrix} -1679 \\ .3578 \\ -.1775 \\ -.0084 \\ .0002 \end{bmatrix} \times 10^{-11}$. The computed solution $\mathbf{x}_1$ is accurate to about 11 decimal places.

**44.** **[M]** Solve $A\mathbf{x} = (0, 0, 0, 0, 1)$. MATLAB shows that $\text{cond}(A) \approx 4.8 \times 10^5$. Since MATLAB computes numbers accurately to 16 decimal places, the entries in the computed value of $\mathbf{x}$ should be accurate to at least 11 digits. The exact solution is (630, –12600, 56700, –88200, 44100).

**45.** **[M]** Some versions of MATLAB issue a warning when asked to invert a Hilbert matrix of order 12 or larger using floating-point arithmetic. The product $AA^{-1}$ should have several off-diagonal entries that are far from being zero. If not, try a larger matrix.

**Note**: All matrix programs supported by the *Study Guide* have data for Exercise 45, but only MATLAB and Maple have a single command to create a Hilbert matrix.

**Notes**: The *Study Guide* for Section 2.3 organizes the statements of the Invertible Matrix Theorem in a table that imbeds these ideas in a broader discussion of rectangular matrices. The statements are arranged in three columns: statements that are logically equivalent for any $m \times n$ matrix and are related to existence concepts, those that are equivalent only for any $n \times n$ matrix, and those that are equivalent for any $n \times p$ matrix and are related to uniqueness concepts. Four statements are included that are not in the text's official list of statements, to give more symmetry to the three columns. You may or may not wish to comment on them.

I believe that students cannot fully understand the concepts in the IMT if they do not know the correct wording of each statement. (Of course, this knowledge is not sufficient for understanding.) The *Study Guide*'s Section 2.3 has an example of the type of question I often put on an exam at this point in the course. The section concludes with a discussion of reviewing and reflecting, as important steps to a mastery of linear algebra.

## 2.4 SOLUTIONS

**Notes**: Partitioned matrices arise in theoretical discussions in essentially every field that makes use of matrices. The *Study Guide* mentions some examples (with references).

Every student should be exposed to some of the ideas in this section. If time is short, you might omit Example 4 and Theorem 10, and replace Example 5 by a problem similar to one in Exercises 1–10. (A sample replacement is given at the end of these solutions.) Then select homework from Exercises 1–13, 15, and 21–24.

The exercises just mentioned provide a good environment for practicing matrix manipulation. Also, students will be reminded that an equation of the form $AB = I$ does not by itself make $A$ or $B$ invertible. (The matrices must be square and the IMT is required.)