Math 260 Homework 2.1

2 Matrix Algebra

2.1 SOLUTIONS

Notes: The definition here of a matrix product AB gives the proper view of AB for nearly all matrix calculations. (The dual fact about the rows of A and the rows of AB is seldom needed, mainly because vectors here are usually written as columns.) I assign Exercise 13 and most of Exercises 17–22 to reinforce the definition of AB.

Exercises 23 and 24 are used in the proof of the Invertible Matrix Theorem, in Section 2.3. Exercises 23–25 are mentioned in a footnote in Section 2.2. A class discussion of the solutions of Exercises 23–25 can provide a transition to Section 2.2. Or, these exercises could be assigned after starting Section 2.2.

Exercises 27 and 28 are optional, but they are mentioned in Example 4 of Section 2.4. Outer products also appear in Exercises 31–34 of Section 4.6 and in the spectral decomposition of a symmetric matrix, in Section 7.1. Exercises 29–33 provide good training for mathematics majors.

1. $-2A=(-2)\begin{bmatrix}2&0&-1\\4&-5&2\end{bmatrix}=\begin{bmatrix}-4&0&2\\-8&10&-4\end{bmatrix}$. Next, use $B-2A=B+(-2A)$:

$$B-2A=\begin{bmatrix}7&-5&1\\1&-4&-3\end{bmatrix}+\begin{bmatrix}-4&0&2\\-8&10&-4\end{bmatrix}=\begin{bmatrix}3&-5&3\\-7&6&-7\end{bmatrix}$$

The product AC is not defined because the number of columns of A does not match the number of rows of C. $CD=\begin{bmatrix}1&2\\-2&1\end{bmatrix}\begin{bmatrix}3&5\\-1&4\end{bmatrix}=\begin{bmatrix}1\cdot3+2(-1)&1\cdot5+2\cdot4\\-2\cdot3+1(-1)&-2\cdot5+1\cdot4\end{bmatrix}=\begin{bmatrix}1&13\\-7&-6\end{bmatrix}$. For mental computation, the row-column rule is probably easier to use than the definition.

2. $A+3B=\begin{bmatrix}2&0&-1\\4&-5&2\end{bmatrix}+3\begin{bmatrix}7&-5&1\\1&-4&-3\end{bmatrix}=\begin{bmatrix}2+21&0-15&-1+3\\4+3&-5-12&2-9\end{bmatrix}=\begin{bmatrix}23&-15&2\\7&-17&-7\end{bmatrix}$

The expression $2C-3E$ is not defined because $2C$ has 2 columns and $-3E$ has only 1 column.

$$DB=\begin{bmatrix}3&5\\-1&4\end{bmatrix}\begin{bmatrix}7&-5&1\\1&-4&-3\end{bmatrix}=\begin{bmatrix}3\cdot7+5\cdot1&3(-5)+5(-4)&3\cdot1+5(-3)\\-1\cdot7+4\cdot1&-1(-5)+4(-4)&-1\cdot1+4(-3)\end{bmatrix}=\begin{bmatrix}26&-35&-12\\-3&-11&-13\end{bmatrix}$$

The product EC is not defined because the number of columns of E does not match the number of rows of C.

3. $3I_2-A=\begin{bmatrix}3&0\\0&3\end{bmatrix}-\begin{bmatrix}2&-5\\3&-2\end{bmatrix}=\begin{bmatrix}3-2&0-(-5)\\0-3&3-(-2)\end{bmatrix}=\begin{bmatrix}1&5\\-3&5\end{bmatrix}$

$(3I_2)A=3(I_2A)=3\begin{bmatrix}2&-5\\3&-2\end{bmatrix}=\begin{bmatrix}6&-15\\9&-6\end{bmatrix}$, or

$$(3I_2)A=\begin{bmatrix}3&0\\0&3\end{bmatrix}\begin{bmatrix}2&-5\\3&-2\end{bmatrix}=\begin{bmatrix}3\cdot2+0&3(-5)+0\\0+3\cdot3&0+3(-2)\end{bmatrix}=\begin{bmatrix}6&-15\\9&-6\end{bmatrix}$$

4. $A-5I_3=\begin{bmatrix}5&-1&3\\-4&3&-6\\-3&1&2\end{bmatrix}-\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}=\begin{bmatrix}0&-1&3\\-4&-2&-6\\-3&1&-3\end{bmatrix}$

$(5I_3)A=5(I_3A)=5A=5\begin{bmatrix}5&-1&3\\-4&3&-6\\-3&1&2\end{bmatrix}=\begin{bmatrix}25&-5&15\\-20&15&-30\\-15&5&10\end{bmatrix}$, or

$$(5I_3)A=\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}\begin{bmatrix}5&-1&3\\-4&3&-6\\-3&1&2\end{bmatrix}$$

$$=\begin{bmatrix}5\cdot5+0+0&5(-1)+0+0&5\cdot3+0+0\\0+5(-4)+0&0+5\cdot3+0&0+5(-6)+0\\0+0+5(-3)&0+0+5\cdot1&0+0+5\cdot2\end{bmatrix}=\begin{bmatrix}25&-5&15\\-20&15&-30\\-15&5&10\end{bmatrix}$$

5. a. $A\mathbf{b}_1=\begin{bmatrix}-1&3\\2&4\\5&-3\end{bmatrix}\begin{bmatrix}4\\-2\end{bmatrix}=\begin{bmatrix}-10\\0\\26\end{bmatrix}$, $A\mathbf{b}_2=\begin{bmatrix}-1&3\\2&4\\5&-3\end{bmatrix}\begin{bmatrix}-2\\3\end{bmatrix}=\begin{bmatrix}11\\8\\-19\end{bmatrix}$

$$AB=\begin{bmatrix}A\mathbf{b}_1&A\mathbf{b}_2\end{bmatrix}=\begin{bmatrix}-10&11\\0&8\\26&-19\end{bmatrix}$$

b. $\begin{bmatrix}-1&3\\2&4\\5&-3\end{bmatrix}\begin{bmatrix}4&-2\\-2&3\end{bmatrix}=\begin{bmatrix}-1\cdot4+3(-2)&-1(-2)+3\cdot3\\2\cdot4+4(-2)&2(-2)+4\cdot3\\5\cdot4-3(-2)&5(-2)-3\cdot3\end{bmatrix}=\begin{bmatrix}-10&11\\0&8\\26&-19\end{bmatrix}$

6. a. $A\mathbf{b}_1=\begin{bmatrix}4&-3\\-3&5\\0&1\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}-5\\12\\3\end{bmatrix}$, $A\mathbf{b}_2=\begin{bmatrix}4&-3\\-3&5\\0&1\end{bmatrix}\begin{bmatrix}4\\-2\end{bmatrix}=\begin{bmatrix}22\\-22\\-2\end{bmatrix}$

$$AB=\begin{bmatrix}A\mathbf{b}_1&A\mathbf{b}_2\end{bmatrix}=\begin{bmatrix}-5&22\\12&-22\\3&-2\end{bmatrix}$$

b. $\begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 4\cdot 1 - 3\cdot 3 & 4\cdot 4 - 3(-2) \\ -3\cdot 1 + 5\cdot 3 & -3\cdot 4 + 5(-2) \\ 0\cdot 1 + 1\cdot 3 & 0\cdot 4 + 1(-2) \end{bmatrix} = \begin{bmatrix} -5 & 22 \\ 12 & -22 \\ 3 & -2 \end{bmatrix}$

7. Since A has 3 columns, B must match with 3 rows. Otherwise, AB is undefined. Since AB has 7 columns, so does B. Thus, B is 3×7.

8. The number of rows of B matches the number of rows of BC, so B has 5 rows.

9. $AB = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 9 \\ -3 & k \end{bmatrix} = \begin{bmatrix} -7 & 18+3k \\ -4 & -9+k \end{bmatrix}$, while $BA = \begin{bmatrix} 1 & 9 \\ -3 & k \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 12 \\ -6-k & -9+k \end{bmatrix}$.

Then $AB = BA$ if and only if $18 + 3k = 12$ and $-4 = -6 - k$, which happens if and only if $k = -2$.

10. $AB = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -21 & -21 \\ 7 & 7 \end{bmatrix}$, $AC = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -3 & -5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -21 & -21 \\ 7 & 7 \end{bmatrix}$

11. $AD = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 6 \\ 10 & 12 & 10 \\ 15 & 15 & 12 \end{bmatrix}$

$DA = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 15 \\ 6 & 12 & 15 \\ 6 & 10 & 12 \end{bmatrix}$

Right-multiplication (that is, multiplication on the right) by the diagonal matrix D multiplies each *column* of A by the corresponding diagonal entry of D. Left-multiplication by D multiplies each *row* of A by the corresponding diagonal entry of D. To make $AB = BA$, one can take B to be a multiple of I_3. For instance, if $B = 4I_3$, then AB and BA are both the same as $4A$.

12. Consider $B = [\mathbf{b}_1 \ \ \mathbf{b}_2]$. To make $AB = 0$, one needs $A\mathbf{b}_1 = \mathbf{0}$ and $A\mathbf{b}_2 = \mathbf{0}$. By inspection of A, a suitable

$\mathbf{b}_1$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, or any multiple of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Example: $B = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$.

13. Use the definition of AB written in reverse order: $[A\mathbf{b}_1 \ \cdots \ A\mathbf{b}_p] = A[\mathbf{b}_1 \ \cdots \ \mathbf{b}_p]$. Thus $[Q\mathbf{r}_1 \ \cdots \ Q\mathbf{r}_p] = QR$, when $R = [\mathbf{r}_1 \ \cdots \ \mathbf{r}_p]$.

14. By definition, $UQ = U[\mathbf{q}_1 \ \cdots \ \mathbf{q}_4] = [U\mathbf{q}_1 \ \cdots \ U\mathbf{q}_4]$. From Example 6 of Section 1.8, the vector $U\mathbf{q}_1$ lists the total costs (material, labor, and overhead) corresponding to the amounts of products B andC specified in the vector $\mathbf{q}_1$. That is, the first column of UQ lists the total costs for materials, labor, and overhead used to manufacture products B and C during the first quarter of the year. Columns 2, 3,and 4 of UQ list the total amounts spent to manufacture B and C during the 2nd, 3rd, and 4th quarters, respectively.

15. **a.** False. See the definition of AB.

b. False. The roles of A and B should be reversed in the second half of the statement. See the box after Example 3.

c. True. See Theorem 2(b), read right to left.

d. True. See Theorem 3(b), read right to left.

e. False. The phrase "in the same order" should be "in the reverse order." See the box after Theorem 3.

16. **a.** True. See the box after Example 4.

b. False. AB must be a 3×3 matrix, but the formula given here implies that it is a 3×1 matrix. The plus signs should just be spaces (between columns). This is a common mistake.

c. True. Apply Theorem 3(d) to $A^2 = AA$

d. False. The left-to-right order of $(ABC)^T$, is $C^TB^TA^T$. The order cannot be changed in general.

e. True. This general statement follows from Theorem 3(b).

17. Since $\begin{bmatrix} -3 & -11 \\ 1 & 17 \end{bmatrix} = AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2]$, the first column of B satisfies the equation $A\mathbf{x} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$. Row reduction: $[A \quad A\mathbf{b}_1] \sim \begin{bmatrix} 1 & -3 & -3 \\ -3 & 5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$. So $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Similarly,

$[A \quad A\mathbf{b}_2] \sim \begin{bmatrix} 1 & -3 & -11 \\ -3 & 5 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Note: An alternative solution of Exercise 17 is to row reduce $[A \ \ A\mathbf{b}_1 \ \ A\mathbf{b}_2]$ with one sequence of row operations. This observation can prepare the way for the inversion algorithm in Section 2.2.

18. The third column of AB is also all zeros because $A\mathbf{b}_3 = A\mathbf{0} = \mathbf{0}$

19. (A solution is in the text). Write $B = [\mathbf{b}_1 \ \ \mathbf{b}_2 \ \ \mathbf{b}_3]$. By definition, the third column of AB is $A\mathbf{b}_3$. By hypothesis, $\mathbf{b}_3 = \mathbf{b}_1 + \mathbf{b}_2$. So $A\mathbf{b}_3 = A(\mathbf{b}_1 + \mathbf{b}_2) = A\mathbf{b}_1 + A\mathbf{b}_2$, by a property of matrix-vector multiplication. Thus, the third column of AB is the sum of the first two columns of AB.

20. The first two columns of AB are $A\mathbf{b}_1$ and $A\mathbf{b}_2$. They are equal since $\mathbf{b}_1$ and $\mathbf{b}_2$ are equal.

21. Let $\mathbf{b}_p$ be the last column of B. By hypothesis, the last column of AB is zero. Thus, $A\mathbf{b}_p = \mathbf{0}$. However, $\mathbf{b}_p$ is not the zero vector, because B has no column of zeros. Thus, the equation $A\mathbf{b}_p = \mathbf{0}$ is a linear dependence relation among the columns of A, and so the columns of A are linearly dependent.

Note: The text answer for Exercise 21 is, "The columns of A are linearly dependent. Why?" The *Study Guide* supplies the argument above, in case a student needs help.

22. If the columns of B are linearly dependent, then there exists a nonzero vector $\mathbf{x}$ such that $B\mathbf{x} = \mathbf{0}$. From this, $A(B\mathbf{x}) = A\mathbf{0}$ and $(AB)\mathbf{x} = \mathbf{0}$ (by associativity). Since $\mathbf{x}$ is nonzero, the columns of AB must be linearly dependent.

23. If $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{0}$, then $CA\mathbf{x} = C\mathbf{0} = \mathbf{0}$ and so $I_n\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \mathbf{0}$. This shows that the equation $A\mathbf{x} = \mathbf{0}$ has no free variables. So every variable is a basic variable and every column of A is a pivot column. (A variation of this argument could be made using linear independence and Exercise 30 in Section 1.7.) Since each pivot is in a different row, A must have at least as many rows as columns.

24. Write $I_3 = [\mathbf{e}_1 \ \ \mathbf{e}_2 \ \ \mathbf{e}_3]$ and $D = [\mathbf{d}_1 \ \ \mathbf{d}_2 \ \ \mathbf{d}_3]$. By definition of AD, the equation $AD = I_3$ is equivalent to the three equations $A\mathbf{d}_1 = \mathbf{e}_1$, $A\mathbf{d}_2 = \mathbf{e}_2$, and $A\mathbf{d}_3 = \mathbf{e}_3$. Each of these equations has at least one solution because the columns of A span $\mathbb{R}^3$. (See Theorem 4 in Section 1.4.) Select one solution of each equation and use them for the columns of D. Then $AD = I_3$.

25. By Exercise 23, the equation $CA = I_n$ implies that (number of rows in A) $\geq$ (number of columns), that is, $m \geq n$. By Exercise 24, the equation $AD = I_m$ implies that (number of rows in A) $\leq$ (number of columns), that is, $m \leq n$. Thus $m = n$. To prove the second statement, observe that $CAD = (CA)D = I_nD = D$, and also $CAD = C(AD) = CI_m = C$. Thus $C = D$. A shorter calculation is

$$C = C\,I_n = C(AD) = (CA)D = I_n\,D = D$$

26. Take any $\mathbf{b}$ in $\mathbb{R}^m$. By hypothesis, $AD\mathbf{b} = I_m\mathbf{b} = \mathbf{b}$. Rewrite this equation as $A(D\mathbf{b}) = \mathbf{b}$. Thus, the vector $\mathbf{x} = D\mathbf{b}$ satisfies $A\mathbf{x} = \mathbf{b}$. This proves that the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$. By Theorem 4 in Section 1.4, A has a pivot position in each row. Since each pivot is in a different column, A must have at least as many columns as rows.

27. The product $\mathbf{u}^T\mathbf{v}$ is a 1×1 matrix, which usually is identified with a real number and is written without the matrix brackets.

$$\mathbf{u}^T\mathbf{v} = \begin{bmatrix}-3 & 2 & -5\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = -3a + 2b - 5c\,,\quad \mathbf{v}^T\mathbf{u} = \begin{bmatrix}a & b & c\end{bmatrix}\begin{bmatrix}-3\\2\\-5\end{bmatrix} = -3a + 2b - 5c$$

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix}-3\\2\\-5\end{bmatrix}\begin{bmatrix}a & b & c\end{bmatrix} = \begin{bmatrix}-3a & -3b & -3c\\ 2a & 2b & 2c\\ -5a & -5b & -5c\end{bmatrix}$$

$$\mathbf{v}\mathbf{u}^T = \begin{bmatrix}a\\b\\c\end{bmatrix}\begin{bmatrix}-3 & 2 & -5\end{bmatrix} = \begin{bmatrix}-3a & 2a & -5a\\ -3b & 2b & -5b\\ -3c & 2c & -5c\end{bmatrix}$$

28. Since the inner product $\mathbf{u}^T\mathbf{v}$ is a real number, it equals its transpose. That is, $\mathbf{u}^T\mathbf{v} = (\mathbf{u}^T\mathbf{v})^T = \mathbf{v}^T(\mathbf{u}^T)^T = \mathbf{v}^T\mathbf{u}$, by Theorem 3(d) regarding the transpose of a product of matrices and by Theorem 3(a). The outer product $\mathbf{u}\mathbf{v}^T$ is an $n\times n$ matrix. By Theorem 3, $(\mathbf{u}\mathbf{v}^T)^T = (\mathbf{v}^T)^T\mathbf{u}^T = \mathbf{v}\mathbf{u}^T$.

29. The (i, j)-entry of $A(B + C)$ equals the (i, j)-entry of $AB + AC$, because

$$\sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}$$

The (i, j)-entry of $(B + C)A$ equals the (i, j)-entry of $BA + CA$, because

$$\sum_{k=1}^{n} (b_{ik} + c_{ik})a_{kj} = \sum_{k=1}^{n} b_{ik}a_{kj} + \sum_{k=1}^{n} c_{ik}a_{kj}$$

30. The (i, j))-entries of $r(AB)$, $(rA)B$, and $A(rB)$ are all equal, because

$$r\sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n} (ra_{ik})b_{kj} = \sum_{k=1}^{n} a_{ik}(rb_{kj})$$

31. Use the definition of the product I_mA and the fact that $I_m\mathbf{x} = \mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^m$.

$$I_mA = I_m[\mathbf{a}_1 \ \cdots \ \mathbf{a}_n] = [I_m\mathbf{a}_1 \ \cdots \ I_m\mathbf{a}_n] = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n] = A$$

32. Let $\mathbf{e}_j$ and $\mathbf{a}_j$ denote the jth columns of I_n and A, respectively. By definition, the jth column of AI_n *is* $A\mathbf{e}_j$, which is simply $\mathbf{a}_j$ because $\mathbf{e}_j$ has 1 in the jth position and zeros elsewhere. Thus corresponding columns of AI_n and A are equal. Hence $AI_n = A$.

33. The (i, j)-entry of $(AB)^T$ is the (j, i)-entry of AB, which is

$$a_{j1}b_{1i} + \cdots + a_{jn}b_{ni}$$

The entries in row i of B^T are $b_{1i}, \ldots, b_{ni}$, because they come from column i of B. Likewise, the entries in column j of A^T are $a_{j1}, \ldots, a_{jn}$, because they come from row j of A. Thus the (i, j)-entry in B^TA^T is $a_{j1}b_{1i} + \cdots + a_{jn}b_{ni}$, as above.

34. Use Theorem 3(d), treating $\mathbf{x}$ as an $n\times 1$ matrix: $(AB\mathbf{x})^T = \mathbf{x}^T(AB)^T = \mathbf{x}^TB^TA^T$.

35. [M] The answer here depends on the choice of matrix program. For MATLAB, use the `help` command to read about `zeros`, `ones`, `eye`, and `diag`. For other programs see the appendices in the *Study Guide*. (The TI calculators have fewer single commands that produce special matrices.)

36. [M] The answer depends on the choice of matrix program. In MATLAB, the command `rand(5,6)` creates a 5×6 matrix with random entries uniformly distributed between 0 and 1. The command

```
round(19*(rand(4,4)-.5))
```

creates a random 4×4 matrix with integer entries between –9 and 9. The same result is produced by the command `randomint` in the Laydata4 Toolbox on text website. For other matrix programs see the appendices in the *Study Guide*.

37. [M] The equality $AB = BA$ is very likely to be false for 4×4 matrices selected at random.

38. [M] $(A + I)(A – I) – (A^2 – I) = 0$ for all 5×5 matrices. However, $(A + B)(A – B) – A^2 – B^2$ is the zero matrix only in the special cases when $AB = BA$. In general,

$$(A + B)(A – B) = A(A – B) + B(A – B) = AA – AB + BA – BB.$$

39. [M] The equality $(A^T+B^T)=(A+B)^T$ and $(AB)^T=B^TA^T$ should always be true, whereas $(AB)^T = A^TB^T$ is very likely to be false for 4×4 matrices selected at random.

40. [M] The matrix S "shifts" the entries in a vector (a, b, c, d, e) to yield $(b, c, d, e, 0)$. The entries in S^2 result from applying S to the columns of S, and similarly for S^3, and so on. This explains the patterns of entries in the powers of S: