

be written in the form $A\mathbf{x} = \mathbf{b}$, where A is determined by the arrangement of the four interior points on the plate and \mathbf{b} is a vector in \mathbf{R}^4 determined by the boundary temperatures.

Note: The MATLAB box in the *Study Guide* for Section 1.10 discusses scientific notation and shows how to generate a matrix whose columns list the vectors $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots$, determined by an equation $\mathbf{x}_{k+1} = M\mathbf{x}_k$ for $k = 0, 1, \dots$

Chapter 1 SUPPLEMENTARY EXERCISES

1. a. False. (The word “reduced” is missing.) Counterexample:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

The matrix A is row equivalent to matrices B and C , both in echelon form.

- b. False. Counterexample: Let A be any $n \times n$ matrix with fewer than n pivot columns. Then the equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions. (Theorem 2 in Section 1.2 says that a system has either zero, one, or infinitely many solutions, but it does not say that a system with infinitely many solutions exists. Some counterexample is needed.)
- c. True. If a linear system has more than one solution, it is a consistent system and has a free variable. By the Existence and Uniqueness Theorem in Section 1.2, the system has infinitely many solutions.
- d. False. Counterexample: The following system has no free variables and no solution:
- $$\begin{aligned} x_1 + x_2 &= 1 \\ x_2 &= 5 \\ x_1 + x_2 &= 2 \end{aligned}$$
- e. True. See the box after the definition of elementary row operations, in Section 1.1. If $[A \ \mathbf{b}]$ is transformed into $[C \ \mathbf{d}]$ by elementary row operations, then the two augmented matrices are row equivalent.
- f. True. Theorem 6 in Section 1.5 essentially says that when $A\mathbf{x} = \mathbf{b}$ is consistent, the solution sets of the nonhomogeneous equation and the homogeneous equation are translates of each other. In this case, the two equations have the same number of solutions.
- g. False. For the columns of A to span \mathbf{R}^m , the equation $A\mathbf{x} = \mathbf{b}$ must be consistent for *all* \mathbf{b} in \mathbf{R}^m , not for just one vector \mathbf{b} in \mathbf{R}^m .
- h. False. Any matrix can be transformed by elementary row operations into reduced echelon form, but not every matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent.
- i. True. If A is row equivalent to B , then A can be transformed by elementary row operations first into B and then further transformed into the reduced echelon form U of B . Since the reduced echelon form of A is unique, it must be U .
- j. False. Every equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution whether or not some variables are free.
- k. True, by Theorem 4 in Section 1.4. If the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every \mathbf{b} in \mathbf{R}^m , then A must have a pivot position in every one of its m rows. If A has m pivot positions, then A has m pivot columns, each containing one pivot position.
- l. False. The word “unique” should be deleted. Let A be any matrix with m pivot columns but more than m columns altogether. Then the equation $A\mathbf{x} = \mathbf{b}$ is consistent and has m basic variables and at least one free variable. Thus the equation does not have a unique solution.

- m. True. If A has n pivot positions, it has a pivot in each of its n columns and in each of its n rows. The reduced echelon form has a 1 in each pivot position, so the reduced echelon form is the $n \times n$ identity matrix.
- n. True. Both matrices A and B can be row reduced to the 3×3 identity matrix, as discussed in the previous question. Since the row operations that transform B into I_3 are reversible, A can be transformed first into I_3 and then into B .
- o. True. The reason is essentially the same as that given for question f.
- p. True. If the columns of A span \mathbf{R}^m , then the reduced echelon form of A is a matrix U with a pivot in each row, by Theorem 4 in Section 1.4. Since B is row equivalent to A , B can be transformed by row operations first into A and then further transformed into U . Since U has a pivot in each row, so does B . By Theorem 4, the columns of B span \mathbf{R}^m .
- q. False. See Example 5 in Section 1.7.
- r. True. Any set of three vectors in \mathbf{R}^2 would have to be linearly dependent, by Theorem 8 in Section 1.7.
- s. False. If a set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ were to span \mathbf{R}^5 , then the matrix $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \mathbf{v}_4]$ would have a pivot position in each of its five rows, which is impossible since A has only four columns.
- t. True. The vector $-\mathbf{u}$ is a linear combination of \mathbf{u} and \mathbf{v} , namely, $-\mathbf{u} = (-1)\mathbf{u} + 0\mathbf{v}$.
- u. False. If \mathbf{u} and \mathbf{v} are multiples, then $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is a line, and \mathbf{w} need not be on that line.
- v. False. Let \mathbf{u} and \mathbf{v} be any linearly independent pair of vectors and let $\mathbf{w} = 2\mathbf{v}$. Then $\mathbf{w} = 0\mathbf{u} + 2\mathbf{v}$, so \mathbf{w} is a linear combination of \mathbf{u} and \mathbf{v} . However, \mathbf{u} cannot be a linear combination of \mathbf{v} and \mathbf{w} because if it were, \mathbf{u} would be a multiple of \mathbf{v} . That is not possible since $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent.
- w. False. The statement would be true if the condition \mathbf{v}_1 is not zero were present. See Theorem 7 in Section 1.7. However, if $\mathbf{v}_1 = \mathbf{0}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, no matter what else might be true about \mathbf{v}_2 and \mathbf{v}_3 .
- x. True. "Function" is another word used for "transformation" (as mentioned in the definition of "transformation" in Section 1.8), and a linear transformation is a special type of transformation.
- y. True. For the transformation $\mathbf{x} \mapsto A\mathbf{x}$ to map \mathbf{R}^5 onto \mathbf{R}^6 , the matrix A would have to have a pivot in every row and hence have six pivot columns. This is impossible because A has only five columns.
- z. False. For the transformation $\mathbf{x} \mapsto A\mathbf{x}$ to be one-to-one, A must have a pivot in each column. Since A has n columns and m pivots, m might be less than n .
2. If $a \neq 0$, then $x = b/a$; the solution is unique. If $a = 0$, and $b \neq 0$, the solution set is empty, because $0x = 0 \neq b$. If $a = 0$ and $b = 0$, the equation $0x = 0$ has infinitely many solutions.
3. a. Any consistent linear system whose echelon form is
- $$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
- b. Any consistent linear system whose coefficient matrix has reduced echelon form I_3 .
- c. Any inconsistent linear system of three equations in three variables.
4. Since there are three pivots (one in each row), the augmented matrix must reduce to the form

$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}$. A solution of $A\mathbf{x} = \mathbf{b}$ exists for all \mathbf{b} because there is a pivot in each row of A . Each solution is unique because there are no free variables.

5. a. $\begin{bmatrix} 1 & 3 & k \\ 4 & h & 8 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & k \\ 0 & h-12 & 8-4k \end{bmatrix}$. If $h = 12$ and $k \neq 2$, the second row of the augmented matrix indicates an inconsistent system of the form $0x_2 = b$, with b nonzero. If $h = 12$, and $k = 2$, there is only one nonzero equation, and the system has infinitely many solutions. Finally, if $h \neq 12$, the coefficient matrix has two pivots and the system has a unique solution.

b. $\begin{bmatrix} -2 & h & 1 \\ 6 & k & -2 \end{bmatrix} \sim \begin{bmatrix} \textcircled{-2} & h & 1 \\ 0 & k+3h & 1 \end{bmatrix}$. If $k + 3h = 0$, the system is inconsistent. Otherwise, the coefficient matrix has two pivots and the system has a unique solution.

6. a. Set $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$. “Determine if \mathbf{b} is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.” Or, “Determine if \mathbf{b} is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.” To do this, compute

$$\begin{bmatrix} 4 & -2 & 7 & -5 \\ 8 & -3 & 10 & -3 \end{bmatrix} \sim \begin{bmatrix} \textcircled{4} & -2 & 7 & -5 \\ 0 & \textcircled{1} & -4 & 7 \end{bmatrix}$$
. The system is consistent, so \mathbf{b} is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

b. Set $A = \begin{bmatrix} 4 & -2 & 7 \\ 8 & -3 & 10 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$. “Determine if \mathbf{b} is a linear combination of the columns of A .”

c. Define $T(\mathbf{x}) = A\mathbf{x}$. “Determine if \mathbf{b} is in the range of T .”

7. a. Set $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -5 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ -5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. “Determine if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span \mathbf{R}^3 .” To do this,

row reduce $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$:

$$\begin{bmatrix} 2 & -4 & -2 \\ -5 & 1 & 1 \\ 7 & -5 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 \\ 0 & -9 & -4 \\ 0 & 9 & 4 \end{bmatrix} \sim \begin{bmatrix} \textcircled{2} & -4 & -2 \\ 0 & \textcircled{-9} & -4 \\ 0 & 0 & 0 \end{bmatrix}$$
. The matrix does not have a pivot in each row,

so its columns do not span \mathbf{R}^3 , by Theorem 4 in Section 1.4.

b. Set $A = \begin{bmatrix} 2 & -4 & -2 \\ -5 & 1 & 1 \\ 7 & -5 & -3 \end{bmatrix}$. “Determine if the columns of A span \mathbf{R}^3 .”

c. Define $T(\mathbf{x}) = A\mathbf{x}$. “Determine if T maps \mathbf{R}^3 onto \mathbf{R}^3 .”

8. a. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * \\ 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$ b. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$

9. The first line is the line spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The second line is spanned by $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. So the problem is to

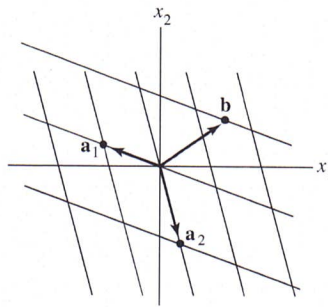
write $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ as the sum of a multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and a multiple of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. That is, find x_1 and x_2 such that

$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$. Reduce the augmented matrix for this equation:

$$\begin{bmatrix} 2 & 1 & 5 \\ 1 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 \\ 2 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 \\ 0 & -3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 7/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4/3 \\ 0 & 1 & 7/3 \end{bmatrix}$$

Thus, $\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \frac{4}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{7}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 8/3 \\ 4/3 \end{bmatrix} + \begin{bmatrix} 7/3 \\ 14/3 \end{bmatrix}$.

10. The line through \mathbf{a}_1 and the origin and the line through \mathbf{a}_2 and the origin determine a “grid” on the x_1x_2 -plane as shown below. Every point in \mathbf{R}^2 can be described uniquely in terms of this grid. Thus, \mathbf{b} can be reached from the origin by traveling a certain number of units in the \mathbf{a}_1 -direction and a certain number of units in the \mathbf{a}_2 -direction.



11. A solution set is a line when the system has one free variable. If the coefficient matrix is 2×3 , then two of the columns should be pivot columns. For instance, take $\begin{bmatrix} 1 & 2 & * \\ 0 & 3 & * \end{bmatrix}$. Put anything in column

3. The resulting matrix will be in echelon form. Make one row replacement operation on the second row to create a matrix *not* in echelon form, such as $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 2 \end{bmatrix}$

12. A solution set is a plane where there are two free variables. If the coefficient matrix is 2×3 , then only one column can be a pivot column. The echelon form will have all zeros in the second row. Use a row replacement to create a matrix not in echelon form. For instance, let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$.

13. The reduced echelon form of A looks like $E = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$. Since E is row equivalent to A , the

equation $E\mathbf{x} = \mathbf{0}$ has the same solutions as $A\mathbf{x} = \mathbf{0}$. Thus $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

By inspection, $E = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

14. Row reduce the augmented matrix for $x_1 \begin{bmatrix} 1 \\ a \end{bmatrix} + x_2 \begin{bmatrix} a \\ a+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (*).

$$\begin{bmatrix} 1 & a & 0 \\ a & a+2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & a & 0 \\ 0 & a+2-a^2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & a & 0 \\ 0 & (2-a)(1+a) & 0 \end{bmatrix}$$

The equation (*) has a nontrivial solution only when $(2-a)(1+a) = 0$. So the vectors are linearly independent for all a except $a = 2$ and $a = -1$.

15. a. If the three vectors are linearly independent, then a , c , and f must all be nonzero. (The converse is true, too.) Let A be the matrix whose columns are the three linearly independent vectors. Then A must have three pivot columns. (See Exercise 30 in Section 1.7, or realize that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and so there can be no free variables in the system of equations.) Since A is 3×3 , the pivot positions are exactly where a , c , and f are located.
- b. The numbers a, \dots, f can have any values. Here's why. Denote the columns by $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 . Observe that \mathbf{v}_1 is not the zero vector. Next, \mathbf{v}_2 is not a multiple of \mathbf{v}_1 because the third entry of \mathbf{v}_2 is nonzero. Finally, \mathbf{v}_3 is not a linear combination of \mathbf{v}_1 and \mathbf{v}_2 because the fourth entry of \mathbf{v}_3 is nonzero. By Theorem 7 in Section 1.7, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
16. Denote the columns from right to left by $\mathbf{v}_1, \dots, \mathbf{v}_4$. The "first" vector \mathbf{v}_1 is nonzero, \mathbf{v}_2 is not a multiple of \mathbf{v}_1 (because the third entry of \mathbf{v}_2 is nonzero), and \mathbf{v}_3 is not a linear combination of \mathbf{v}_1 and \mathbf{v}_2 (because the second entry of \mathbf{v}_3 is nonzero). Finally, by looking at first entries in the vectors, \mathbf{v}_4 cannot be a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 . By Theorem 7 in Section 1.7, the columns are linearly independent.
17. Here are two arguments. The first is a "direct" proof. The second is called a "proof by contradiction."
- i. Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set, $\mathbf{v}_1 \neq \mathbf{0}$. Also, Theorem 7 shows that \mathbf{v}_2 cannot be a multiple of \mathbf{v}_1 , and \mathbf{v}_3 cannot be a linear combination of \mathbf{v}_1 and \mathbf{v}_2 . By hypothesis, \mathbf{v}_4 is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 . Thus, by Theorem 7, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ cannot be a linearly dependent set and so must be linearly independent.
- ii. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent. Then by Theorem 7, one of the vectors in the set is a linear combination of the preceding vectors. This vector cannot be \mathbf{v}_4 because \mathbf{v}_4 is *not* in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Also, none of the vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linear combination of the preceding vectors, by Theorem 7. So the linear dependence of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is impossible. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent.

18. Suppose that c_1 and c_2 are constants such that

$$c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{0} \quad (*)$$

Then $(c_1 + c_2)\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$. Since \mathbf{v}_1 and \mathbf{v}_2 are linearly independent, both $c_1 + c_2 = 0$ and $c_2 = 0$. It follows that both c_1 and c_2 in (*) must be zero, which shows that $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2\}$ is linearly independent.

19. Let M be the line through the origin that is parallel to the line through $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 . Then $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$ are both on M . So one of these two vectors is a multiple of the other, say $\mathbf{v}_2 - \mathbf{v}_1 = k(\mathbf{v}_3 - \mathbf{v}_1)$. This equation produces a linear dependence relation $(k-1)\mathbf{v}_1 + \mathbf{v}_2 - k\mathbf{v}_3 = \mathbf{0}$.

A second solution: A parametric equation of the line is $\mathbf{x} = \mathbf{v}_1 + t(\mathbf{v}_2 - \mathbf{v}_1)$. Since \mathbf{v}_3 is on the line, there is some t_0 such that $\mathbf{v}_3 = \mathbf{v}_1 + t_0(\mathbf{v}_2 - \mathbf{v}_1) = (1 - t_0)\mathbf{v}_1 + t_0\mathbf{v}_2$. So \mathbf{v}_3 is a linear combination of \mathbf{v}_1 and \mathbf{v}_2 , and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

20. If $T(\mathbf{u}) = \mathbf{v}$, then since T is linear,

$$T(-\mathbf{u}) = T((-1)\mathbf{u}) = (-1)T(\mathbf{u}) = -\mathbf{v}.$$

21. Either compute $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$, and $T(\mathbf{e}_3)$ to make the columns of A , or write the vectors vertically in the definition of T and fill in the entries of A by inspection:

$$A\mathbf{x} = \begin{bmatrix} ? & ? & ? \\ ? & A & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

22. By Theorem 12 in Section 1.9, the columns of A span \mathbf{R}^3 . By Theorem 4 in Section 1.4, A has a pivot in each of its three rows. Since A has three columns, each column must be a pivot column. So the equation $A\mathbf{x} = \mathbf{0}$ has no free variables, and the columns of A are linearly independent. By Theorem 12 in Section 1.9, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

23. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ implies that $\begin{cases} 4a - 3b = 5 \\ 3a + 4b = 0 \end{cases}$. Solve:

$$\begin{bmatrix} 4 & -3 & 5 \\ 3 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 4 & -3 & 5 \\ 0 & 25/4 & -15/4 \end{bmatrix} \sim \begin{bmatrix} 4 & -3 & 5 \\ 0 & 1 & -3/5 \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & 16/5 \\ 0 & 1 & -3/5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4/5 \\ 0 & 1 & -3/5 \end{bmatrix}$$

Thus $a = 4/5$ and $b = -3/5$.

24. The matrix equation displayed gives the information $2a - 4b = 2\sqrt{5}$ and $4a + 2b = 0$. Solve for a and

$$b: \begin{bmatrix} 2 & -4 & 2\sqrt{5} \\ 4 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 2\sqrt{5} \\ 0 & 10 & -4\sqrt{5} \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & \sqrt{5} \\ 0 & 1 & -2/\sqrt{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/\sqrt{5} \\ 0 & 1 & -2/\sqrt{5} \end{bmatrix}$$

So $a = 1/\sqrt{5}$, $b = -2/\sqrt{5}$.

25. a. The vector lists the number of three-, two-, and one-bedroom apartments provided when x_1 floors of plan A are constructed.

$$b. \quad x_1 \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 3 \\ 9 \end{bmatrix}$$

$$c. \quad [M] \text{ Solve } x_1 \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 66 \\ 74 \\ 136 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 5 & 66 \\ 7 & 4 & 3 & 74 \\ 8 & 8 & 9 & 136 \end{bmatrix} \sim \dots \begin{bmatrix} 1 & 0 & -1/2 & 2 \\ 0 & 1 & 13/8 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 - (1/2)x_3 = 2 \\ x_2 + (13/8)x_3 = 15 \\ 0 = 0 \end{array}$$

The general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 + (1/2)x_3 \\ 15 - (13/8)x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1/2 \\ -13/8 \\ 1 \end{bmatrix}$$

However, the only feasible solutions must have whole numbers of floors for each plan. Thus, x_3 must be a multiple of 8, to avoid fractions. One solution, for $x_3 = 0$, is to use 2 floors of plan A and 15 floors of plan B. Another solution, for $x_3 = 8$, is to use 6 floors of plan A, 2 floors of plan B, and 8 floors of plan C. These are the only feasible solutions. A larger positive multiple of 8 for x_3 makes x_2 negative. A negative value for x_3 , of course, is not feasible either.

