37. [M] $\begin{bmatrix} 2 & 3 & 5 & -5 & 0 \\ -7 & 7 & 0 & 0 & 0 \\ -3 & 4 & 1 & 3 & 0 \\ -9 & 3 & -6 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \end{cases}$ $\mathbf{x} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

38. [M] $\begin{bmatrix} 3 & 4 & -7 & 0 & 0 \\ 5 & -8 & 7 & 4 & 0 \\ 6 & -8 & 6 & 4 & 0 \\ 9 & -7 & -2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{cases} x_1 = -x_4 \\ x_2 = -x_4 \\ x_3 = -x_4 \\ x_4 \text{ is free} \end{cases}$ $\mathbf{x} = x_4 \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

39. [M] $\begin{bmatrix} 2 & 3 & 5 & -5 & 8 \\ -7 & 7 & 0 & 0 & 7 \\ -3 & 4 & 1 & 3 & 5 \\ -9 & 3 & -6 & -4 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, yes, **b** is in the range of the transformation,

because the augmented matrix shows a consistent system. In fact,

the general solution is $\begin{cases} x_1 = 1 - x_3 \\ x_2 = 2 - x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \end{cases}$; when $x_3 = 0$ a solution is $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

40. [M] $\begin{bmatrix} 3 & 4 & -7 & 0 & 4 \\ 5 & -8 & 7 & 4 & -4 \\ 6 & -8 & 6 & 4 & -4 \\ 9 & -7 & -2 & 0 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, yes, **b** is in the range of the transformation,

because the augmented matrix shows a consistent system. In fact,

the general solution is $\begin{cases} x_1 = 1 - x_4 \\ x_2 = 2 - x_4 \\ x_3 = 1 - x_4 \\ x_4 \text{ is free} \end{cases}$; when $x_4 = 0$ a solution is $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$.

Notes: At the end of Section 1.8, the *Study Guide* provides a list of equations, figures, examples, and connections with concepts that will strengthen a student's understanding of linear transformations. I encourage my students to continue the construction of review sheets similar to those for "span" and "linear independence," but I refrain from collecting these sheets. At some point the students have to assume the responsibility for mastering this material.

If your students are using MATLAB or another matrix program, you might insert the definition of matrix multiplication after this section, and then assign a project that uses random matrices to explore properties of matrix multiplication. See Exercises 34–36 in Section 2.1. Meanwhile, in class you can continue with your plans for finishing Chapter 1. When you get to Section 2.1, you won't have much to do. The Study Guide's MATLAB note for Section 2.1 contains the matrix notation students will need for a project on matrix multiplication. The appendices in the Study Guide have the corresponding material for Mathematica, Maple, and the TI-83+/84+/89 calculators.

1.9 SOLUTIONS

Notes: This section is optional if you plan to treat linear transformations only lightly, but many instructors will want to cover at least Theorem 10 and a few geometric examples. Exercises 15 and 16 illustrate a fast way to solve Exercises 17–22 without explicitly computing the images of the standard basis.

The purpose of introducing *one-to-one* and *onto* is to prepare for the term *isomorphism* (in Section 4.4) and to acquaint math majors with these terms. Mastery of these concepts would require a substantial digression, and some instructors prefer to omit these topics (and Exercises 25–40). In this case, you can use the result of Exercise 31 in Section 1.8 to show that the coordinate mapping from a vector space onto $\mathbf{R}^n$ (in Section 4.4) preserves linear independence and dependence of sets of vectors. (See Example 6 in Section 4.4.) The notions of one-to-one and onto appear in the Invertible Matrix Theorem (Section 2.3), but can be omitted there if desired

Exercises 25–28 and 31–36 offer fairly easy writing practice. Exercises 31, 32, and 35 provide important links to earlier material.

1. $A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2)] = \begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$

2. $A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad T(\mathbf{e}_3)] = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{bmatrix}$

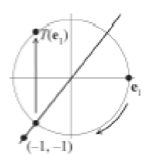
3. $T(\mathbf{e}_1) = \mathbf{e}_1 - 3\mathbf{e}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $T(\mathbf{e}_2) = \mathbf{e}_2$, $A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

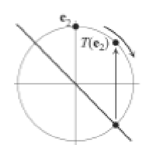
4. $T(\mathbf{e}_1) = \mathbf{e}_1$, $T(\mathbf{e}_2) = \mathbf{e}_2 + 2\mathbf{e}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

5. $T(\mathbf{e}_1) = \mathbf{e}_2$, $T(\mathbf{e}_2) = -\mathbf{e}_1$. $A = [\mathbf{e}_2 \quad -\mathbf{e}_1] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

6. $T(\mathbf{e}_1) = \mathbf{e}_2$, $T(\mathbf{e}_2) = -\mathbf{e}_1$. $A = [\mathbf{e}_2 \quad -\mathbf{e}_1] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

7. Follow what happens to $\mathbf{e}_1$ and $\mathbf{e}_2$. Since $\mathbf{e}_1$ is on the unit circle in the plane, it rotates through $-3\pi/4$ radians into a point on the unit circle that lies in the third quadrant and on the line $x_2 = x_1$ (that is, $y = x$ in more familiar notation). The point $(-1,-1)$ is on the ine $x_2 = x_1$, but its distance from the origin is $\sqrt{2}$. So the rotational image of $\mathbf{e}_1$ is $(-1/\sqrt{2}, -1/\sqrt{2})$. Then this image reflects in the horizontal axis to $(-1/\sqrt{2}, 1/\sqrt{2})$. Similarly, $\mathbf{e}_2$ rotates into a point on the unit circle that lies in the second quadrant and on the





line $x_2 = -x_1$, namely, $(1/\sqrt{2}, -1/\sqrt{2})$. Then this image reflects in the horizontal axis to $(1/\sqrt{2}, 1/\sqrt{2})$. When the two calculations described above are written in vertical vector notation, the transformation's standard matrix $[T(\mathbf{e}_1) \quad T(\mathbf{e}_2)]$ is easily seen:

$$\mathbf{e}_1 \to \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \to \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \ \mathbf{e}_2 \to \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \to \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \ A = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

8. The horizontal shear maps $\mathbf{e}_1$ into $\mathbf{e}_1$, and then the reflection in the line $x_2 = -x_1$ maps $\mathbf{e}_1$ into $-\mathbf{e}_2$. (See Table 1.) The horizontal shear maps $\mathbf{e}_2$ into $\mathbf{e}_2$ into $\mathbf{e}_2 + 2\mathbf{e}_1$. To find the image of $\mathbf{e}_2 + 2\mathbf{e}_1$ when it is reflected in the line $x_2 = -x_1$, use the fact that such a reflection is a linear transformation. So, the image of $\mathbf{e}_2 + 2\mathbf{e}_1$ is the same linear combination of the images of $\mathbf{e}_2$ and $\mathbf{e}_1$, namely, $-\mathbf{e}_1 + 2(-\mathbf{e}_2) = -\mathbf{e}_1 - 2\mathbf{e}_2$. To summarize,

$$\mathbf{e}_1 \to \mathbf{e}_1 \to -\mathbf{e}_2 \text{ and } \mathbf{e}_2 \to \mathbf{e}_2 + 2\mathbf{e}_1 \to -\mathbf{e}_1 - 2\mathbf{e}_2, \text{ so } A = \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix}$$

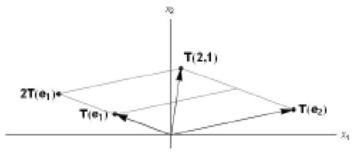
9. $\mathbf{e}_1 \to \mathbf{e}_1 \to -\mathbf{e}_2$ and $\mathbf{e}_2 \to -\mathbf{e}_2 \to -\mathbf{e}_1$, so $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

10. $\mathbf{e}_1 \to \mathbf{e}_1 \to \mathbf{e}_2$ and $\mathbf{e}_2 \to -\mathbf{e}_2 \to -\mathbf{e}_1$, so $A = [\mathbf{e}_2 \quad -\mathbf{e}_1] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

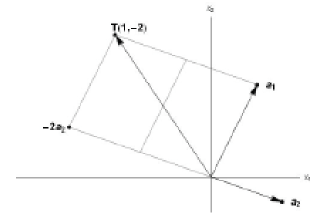
11. The transformation T described maps $\mathbf{e}_1 \to \mathbf{e}_1 \to -\mathbf{e}_1$ and maps $\mathbf{e}_2 \to -\mathbf{e}_2 \to -\mathbf{e}_2$. A rotation through π radians also maps $\mathbf{e}_1$ into $-\mathbf{e}_1$ and maps $\mathbf{e}_2$ into $-\mathbf{e}_2$. Since a linear transformation is completely determined by what it does to the columns of the identity matrix, the rotation transformation has the same effect as T on every vector in $\mathbb{R}^2$.

12. The transformation T in Exercise 10 maps $\mathbf{e}_1 \to \mathbf{e}_1 \to \mathbf{e}_2$ and maps $\mathbf{e}_2 \to -\mathbf{e}_2 \to -\mathbf{e}_1$. A rotation about the origin through $\pi/2$ radians also maps $\mathbf{e}_1$ into $\mathbf{e}_2$ and maps $\mathbf{e}_2$ into $-\mathbf{e}_1$. Since a linear transformation is completely determined by what it does to the columns of the identity matrix, the rotation transformation has the same effect as T on every vector in $\mathbb{R}^2$.

13. Since (2, 1)=2 $\mathbf{e}_1$+ $\mathbf{e}_2$, the image of (2, 1) under T is $2T(\mathbf{e}_1) + T(\mathbf{e}_2)$, by linearity of T. On the figure in the exercise, locate $2T(\mathbf{e}_1)$ and use it with $T(\mathbf{e}_2)$ to form the parallelogram shown below.



14. Since $T(\mathbf{x}) = A\mathbf{x} = [\mathbf{a}_1 \quad \mathbf{a}_2]\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{a}_1 - 2\mathbf{a}_2$, when $\mathbf{x} = (1, -2)$, the image of $\mathbf{x}$ is located by forming the parallelogram shown below.



15. By inspection, $\begin{bmatrix} 2 & -4 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 4x_2 \\ x_1 - x_3 \\ -x_2 + 3x_3 \end{bmatrix}$

16. By inspection, $\begin{bmatrix} 3 & -2 \\ 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_2 \\ x_1 + 4x_2 \\ x_2 \end{bmatrix}$

17. To express $T(\mathbf{x})$ as $A\mathbf{x}$, write $T(\mathbf{x})$ and $\mathbf{x}$ as column vectors, and then fill in the entries in A by inspection, as done in Exercises 15 and 16. Note that since $T(\mathbf{x})$ and $\mathbf{x}$ have four entries, A must be a 4×4 matrix.

$$T(\mathbf{x}) = \begin{bmatrix} x_1 + 2x_2 \\ 0 \\ 2x_2 + x_4 \\ x_2 - x_4 \end{bmatrix} = \begin{bmatrix} & & & \\ & A & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

18. As in Exercise 17, write $T(\mathbf{x})$ and $\mathbf{x}$ as column vectors. Since $\mathbf{x}$ has 2 entries, A has 2 columns. Since $T(\mathbf{x})$ has 4 entries, A has 4 rows.

$$\begin{bmatrix} x_1 + 4x_2 \\ 0 \\ x_1 - 3x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} & \\ & A \\ & \\ & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 0 \\ 1 & -3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

19. Since $T(\mathbf{x})$ has 2 entries, A has 2 rows. Since $\mathbf{x}$ has 3 entries, A has 3 columns.

$$\begin{bmatrix} x_1 - 5x_2 + 4x_3 \\ x_2 - 6x_3 \end{bmatrix} = \begin{bmatrix} & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

20. Since $T(\mathbf{x})$ has 1 entry, A has 1 row. Since $\mathbf{x}$ has 4 entries, A has 4 columns.

$$[3x_1 + 4x_3 - 2x_4] = [\quad A \quad] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = [3 \quad 0 \quad 4 \quad -2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

21. $T(\mathbf{x}) = \begin{bmatrix} x_1+x_2 \\ 4x_1+5x_2 \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. To solve $T(\mathbf{x}) = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$, row reduce the augmented matrix: $\begin{bmatrix} 1 & 1 & 3 \\ 4 & 5 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 7 \\ 0 & \textcircled{1} & -4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$.

22. $T(\mathbf{x}) = \begin{bmatrix} 2x_1 - x_2 \\ -3x_1 + x_2 \\ 2x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. To solve $T(\mathbf{x}) = \begin{bmatrix} 0 \\ -1 \\ -4 \end{bmatrix}$, row reduce the augmented matrix:

$$\begin{bmatrix} 2 & -1 & 0 \\ -3 & 1 & -1 \\ 2 & -3 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 \\ 0 & -1/2 & -1 \\ 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 1 \\ 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

23. **a.** True. See Theorem 10.
b. True. See Example 3.
c. False. See the paragraph before Table 1.
d. False. See the definition of *onto*. *Any* function from $\mathbf{R}^n$ to $\mathbf{R}^m$ maps each vector onto another vector.
e. False. See Example 5.

24. **a.** False. See Theorem 12.
b. True. See Theorem 10.
c. True. See Theorem 10.
d. False. See the definition of one-to-one. Any *function* from $\mathbf{R}^n$ to $\mathbf{R}^m$ maps a vector onto a single (unique) vector.
e. False. See Table 3.

25. A row interchange and a row replacement on the standard matrix A of the transformation T in Exercise 17 produce $\begin{bmatrix} \textcircled{1} & 2 & 0 & 0 \\ 0 & \textcircled{1} & 0 & -1 \\ 0 & 0 & 0 & \textcircled{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. This matrix shows that A has only three pivot positions, so the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. By Theorem 11, the transformation T is *not* one-to-one. Also, since A does not have a pivot in each row, the columns of A do not span $\mathbf{R}^4$. By Theorem 12, T does *not* map $\mathbf{R}^4$ onto $\mathbf{R}^4$.

26. The standard matrix A of the transformation T in Exercise 2 is 2×3. Its columns are linearly dependent because A has more columns than rows. So T is *not* one-to-one, by Theorem 12. Also, A is row equivalent to $\begin{bmatrix} \textcircled{1} & -2 & 3 \\ 0 & \textcircled{17} & -20 \end{bmatrix}$, which shows that the rows of A span $\mathbf{R}^2$. By Theorem 12, T maps $\mathbf{R}^3$ onto $\mathbf{R}^2$.

27. The standard matrix A of the transformation T in Exercise 19 is $\begin{bmatrix} \textcircled{1} & -5 & 4 \\ 0 & \textcircled{1} & -6 \end{bmatrix}$. The columns of A are linearly dependent because A has more columns than rows. So T is *not* one-to-one, by Theorem 12. Also, A has a pivot in each row, so the rows of A span $\mathbf{R}^2$. By Theorem 12, T maps $\mathbf{R}^3$ onto $\mathbf{R}^2$.

28. The standard matrix A of the transformation T in Exercise 14 has linearly independent columns, because the figure in that exercise shows that $\mathbf{a}_1$ and $\mathbf{a}_2$ are not multiples. So T is one-to-one, by Theorem 12. Also, A must have a pivot in each column because the equation $A\mathbf{x} = \mathbf{0}$ has no free variables. Thus, the echelon form of A is $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}$. Since A has a pivot in each row, the columns of A span $\mathbf{R}^2$. So T maps $\mathbf{R}^2$ onto $\mathbf{R}^2$. An alternate argument for the second part is to observe directly from the figure in Exercise 14 that $\mathbf{a}_1$ and $\mathbf{a}_2$ span $\mathbf{R}^2$. This is more or less evident, based on experience with grids such as those in Figure 8 and Exercise 7 of Section 1.3.

29. By Theorem 12, the columns of the standard matrix A must be linearly independent and hence the equation $A\mathbf{x} = \mathbf{0}$ has no free variables. So each column of A must be a pivot column:
$A \sim \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$. Note that T cannot be onto because of the shape of A.

30. By Theorem 12, the columns of the standard matrix A must span $\mathbb{R}^3$. By Theorem 4, the matrix must have a pivot in each row. There are four possibilities for the echelon form:

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

Note that T cannot be one-to-one because of the shape of A.

31. "T is one-to-one if and only if A has n pivot columns." By Theorem 12(b), T is one-to-one if and only if the columns of A are linearly independent. And from the statement in Exercise 30 in Section 1.7, the columns of A are linearly independent if and only if A has n pivot columns.

32. The transformation T maps $\mathbf{R}^n$ onto $\mathbf{R}^m$ if and only if the columns of A span $\mathbf{R}^m$, by Theorem 12. This happens if and only if A has a pivot position in each row, by Theorem 4 in Section 1.4. Since A has m rows, this happens if and only if A has m pivot columns. Thus, "T maps $\mathbf{R}^n$ onto $\mathbf{R}^m$ if and only A has m pivot columns."

33. Define $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathbf{x}) = B\mathbf{x}$ for some $m \times n$ matrix B, and let A be the standard matrix for T. By definition, $A = [T(\mathbf{e}_1) \ \cdots \ T(\mathbf{e}_n)]$, where $\mathbf{e}_j$ is the jth column of I_n. However, by matrix-vector multiplication, $T(\mathbf{e}_j) = B\mathbf{e}_j = \mathbf{b}_j$, the jth column of B. So $A = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_n] = B$.

34. Take $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^p$ and let c and d be scalars. Then

$T(S(c\mathbf{u} + d\mathbf{v})) = T(c \cdot S(\mathbf{u}) + d \cdot S(\mathbf{v}))$ because S is linear
$= c \cdot T(S(\mathbf{u})) + d \cdot T(S(\mathbf{v}))$ because T is linear

This calculation shows that the mapping $\mathbf{x} \to T(S(\mathbf{x}))$ is linear. See equation (4) in Section 1.8.

35. If $T:\mathbb{R}^n \to \mathbb{R}^m$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$, then its standard matrix A has a pivot in each row, by Theorem 12 and by Theorem 4 in Section 1.4. So A must have at least as many columns as rows. That is, $m \le n$. When T is one-to-one, A must have a pivot in each column, by Theorem 12, so $m \ge n$.

36. The transformation T maps $\mathbf{R}^n$ *onto* $\mathbf{R}^m$ if and only if for each $\mathbf{y}$ in $\mathbf{R}^m$ *there exists* an $\mathbf{x}$ in $\mathbf{R}^n$ such that $\mathbf{y} = T(\mathbf{x})$.

37. **[M]** $\begin{bmatrix} -5 & 6 & -5 & -6 \\ 8 & 3 & -3 & 8 \\ 2 & 9 & 5 & -12 \\ -3 & 2 & 7 & -12 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 1 \\ 0 & \textcircled{1} & 0 & -1 \\ 0 & 0 & \textcircled{1} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. There is no pivot in the fourth column of the standard matrix A, so the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution. By Theorem 11, the transformation T is *not* one-to-one. (For a shorter argument, use the result of Exercise 31.)

38. **[M]** $\begin{bmatrix} 7 & 5 & 9 & -9 \\ 5 & 6 & 4 & -4 \\ 4 & 8 & 0 & 7 \\ -6 & -6 & 6 & 5 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 0 \\ 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$. Yes. There is a pivot in every column of the standard matrix A, so the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. By Theorem 11, the transformation T is one-to-one. (For a shorter argument, use the result of Exercise 31.)

39. **[M]** $\begin{bmatrix} 4 & -7 & 3 & 7 & 5 \\ 6 & -8 & 5 & 12 & -8 \\ -7 & 10 & -8 & -9 & 14 \\ 3 & -5 & 4 & 2 & -6 \\ -5 & 6 & -6 & -7 & 3 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 5 & 0 \\ 0 & \textcircled{1} & 0 & 1 & 0 \\ 0 & 0 & \textcircled{1} & -2 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. There is not a pivot in every row, so the columns of the standard matrix do not span $\mathbf{R}^5$. By Theorem 12, the transformation T does *not* map $\mathbf{R}^5$ onto $\mathbf{R}^5$.

40. **[M]** $\begin{bmatrix} 9 & 43 & 5 & 6 & -1 \\ 14 & 15 & -7 & -5 & 4 \\ -8 & -6 & 12 & -5 & -9 \\ -5 & -6 & -4 & 9 & 8 \\ 13 & 14 & 15 & 3 & 11 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 0 & 0 \\ 0 & \textcircled{1} & 0 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$. There is a pivot in every row, so the columns of the standard matrix span $\mathbf{R}^5$. By Theorem 12, the transformation T maps $\mathbf{R}^5$ onto $\mathbf{R}^5$.