Math 260 Homework 1.6

1. Fill in the exchange table one column at a time. The entries in a column describe where a sector's output goes. The decimal fractions in each column sum to 1.

	Distribution of Output From:			
	Goods	Services		Purchased by:
output	↓	↓	input	
	.2	.7	→	Goods
	.8	.3	→	Services

Denote the total annual output (in dollars) of the sectors by p_G and p_S. From the first row, the total input to the Goods sector is $.2\,p_G + .7\,p_S$. The Goods sector must pay for that. So the equilibrium prices must satisfy

$$\begin{array}{ccc} \text{income} & & \text{expenses} \\ p_G & = & .2p_G + .7p_S \end{array}$$

From the second row, the input (that is, the expense) of the Services sector is $.8\,p_G + .3\,p_S$. The equilibrium equation for the Services sector is

$$\begin{array}{ccc} \text{income} & & \text{expenses} \\ p_S & = & .8p_G + .3p_S \end{array}$$

Move all variables to the left side and combine like terms:

$$\begin{array}{rcrcl} .8p_G & - & .7p_S & = & 0 \\ -.8p_G & + & .7p_S & = & 0 \end{array}$$

Row reduce the augmented matrix:

$$\begin{bmatrix} .8 & -.7 & 0 \\ -.8 & .7 & 0 \end{bmatrix} \sim \begin{bmatrix} .8 & -.7 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -.875 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution is $p_G = .875\,p_S$, with p_S free. One equilibrium solution is $p_S = 1000$ and $p_G = 875$. If one uses fractions instead of decimals in the calculations, the general solution would be written $p_G = (7/8)\,p_S$, and a natural choice of prices might be $p_S = 80$ and $p_G = 70$. Only the *ratio* of the prices is important: $p_G = .875\,p_S$. The economic equilibrium is unaffected by a proportional change in prices.

2. Take some other value for p_S, say 200 million dollars. The other equilibrium prices are then $p_C = 188$ million, $p_E = 170$ million. Any constant nonnegative multiple of these prices is a set of equilibrium prices, because the solution set of the system of equations consists of all multiples of one vector. Changing the unit of measurement to another currency such as Japanese yen has the same effect as multiplying all equilibrium prices by a constant. The *ratios* of the prices remain the same, no matter what currency is used.

3. **a.** Fill in the exchange table one column at a time. The entries in a column describe where a sector's output goes. The decimal fractions in each column sum to 1.

	Distribution of Output From:				
	Fuels and Power	Manufacturing	Services		Purchased by:
output	↓	↓	↓	input	
	.10	.10	.20	→	Fuels and Power
	.80	.10	.40	→	Manufacturing
	.10	.80	.40	→	Services

b. Denote the total annual output (in dollars) of the sectors by p_F, p_M, and p_S. From the first row of the table, the total input to the Fuels & Power sector is $.1p_F + .1p_M + .2p_S$. So the equilibrium prices must satisfy

$$\begin{array}{ccc} \text{income} & & \text{expenses} \\ p_F & = & .1p_F + .1p_M + .2p_S \end{array}$$

From the second and third rows of the table, the income/expense requirements for the Manufacturing sector and the Services sector are, respectively,

$$p_M = .8p_F + .1p_M + .4p_S$$
$$p_S = .1p_F + .8p_M + .4p_S$$

Move all variables to the left side and combine like terms:

$$\begin{aligned} .9p_F - .1p_M - .2p_S &= 0 \\ -.8p_F + .9p_M - .4p_S &= 0 \\ -.1p_F - .8p_M + .6p_S &= 0 \end{aligned} \qquad \begin{bmatrix} .9 & -.1 & -.2 & 0 \\ -.8 & .9 & -.4 & 0 \\ -.1 & -.8 & .6 & 0 \end{bmatrix}$$

c. [M] You can obtain the reduced echelon form with a matrix program.

$$\begin{bmatrix} .9 & -.1 & -.2 & 0 \\ -.8 & .9 & -.4 & 0 \\ -.1 & -.8 & .6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -.301 & 0 \\ 0 & 1 & -.712 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The number of decimal places displayed is somewhat arbitrary.

The general solution is $p_F = .301\,p_S$, $p_M = .712\,p_S$, with p_S free. If p_S is assigned the value of 100, then $p_F = 30.1$ and $p_M = 71.2$. Note that only the *ratios* of the prices are determined. This makes sense, for if they were converted from, say, dollars to yen or Euros, the inputs and outputs of each sector would still balance. The economic equilibrium is not affected by a proportional change in prices.

4. a. Fill in the exchange table one column at a time. The entries in each column must sum to 1.

Distribution of Output From:

	Mining	Lumber	Energy	Transportation		Purchased by:
output	↓	↓	↓	↓	input	
	.30	.15	.20	.20	→	Mining
	.10	.15	.15	.10	→	Lumber
	.60	.50	.45	.50	→	Energy
	0	.20	.20	.20	→	Transportation

b. [M] Denote the total annual output of the sectors by p_M, p_L, p_E, and p_T, respectively. From the first row of the table, the total input to Agriculture is $.30p_M + .15p_L + .20p_E + .20\,p_T$. So the equilibrium prices must satisfy

income   expenses

$$p_M = .30p_M + .15p_L + .20p_E + .20p_T$$

From the second, third, and fourth rows of the table, the equilibrium equations are

$$\begin{aligned} p_L &= .10p_M + .15p_L + .15p_E + .10p_T \\ p_E &= .60p_M + .50p_L + .45p_E + .50p_T \\ p_T &= .20p_L + .20p_E + .20p_T \end{aligned}$$

Move all variables to the left side and combine like terms:

$$\begin{aligned} .70p_M - .15p_L - .20p_E - .20p_T &= 0 \\ -.10p_M + .85p_L - .15p_E - .10p_T &= 0 \\ -.60p_M - .50p_L + .55p_E - .50p_T &= 0 \\ -.20p_L - .20p_E + .80p_T &= 0 \end{aligned}$$

Reduce the augmented matrix to reduced echelon form:

$$\begin{bmatrix} .70 & -.15 & -.20 & -.20 & 0 \\ -.10 & .85 & -.15 & -.10 & 0 \\ -.60 & -.50 & .55 & -.50 & 0 \\ 0 & -.20 & -.20 & .80 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1.37 & 0 \\ 0 & 1 & 0 & -.84 & 0 \\ 0 & 0 & 1 & -3.16 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solve for the basic variables in terms of the free variable p_T, and obtain $p_M = 1.37p_T$, $p_L = .84p_T$, and $p_E = 3.16p_T$. The data probably justifies at most two significant figures, so take $p_T = 100$ and round off the other prices to $p_M = 137$, $p_L = 84$, and $p_E = 316$.

5. a. Fill in the exchange table one column at a time. The entries in each column must sum to 1.

Distribution of Output From:

	Agriculture	Manufacturing	Services	Transportation		Purchased by:
output	↓	↓	↓	↓	input	
	.20	.35	.10	.20	→	Agriculture
	.20	.10	.20	.30	→	Manufacturing
	.30	.35	.50	.20	→	Services
	.30	.20	.20	.30	→	Transportation

b. [M] Denote the total annual output of the sectors by p_A, p_M, p_S, and p_T, respectively. The equilibrium equations are

$$\begin{aligned} p_A &= .20p_A + .35p_M + .10p_S + .20p_T \\ p_M &= .20p_A + .10p_M + .20p_S + .30p_T \\ p_S &= .30p_A + .35p_M + .50p_S + .20p_T \\ p_T &= .30p_A + .20p_M + .20p_S + .30p_T \end{aligned}$$

Move all variables to the left side and combine like terms:

$$\begin{aligned} .80p_A - .35p_M - .10p_S - .20p_T &= 0 \\ -.20p_A + .90p_M - .20p_S - .30p_T &= 0 \\ -.30p_A - .35p_M + .50p_S - .20p_T &= 0 \\ -.30p_A - .20p_M - .20p_S + .70p_T &= 0 \end{aligned}$$

Reduce the augmented matrix to reduced echelon form:

$$\begin{bmatrix} .80 & -.35 & -.10 & -.20 & 0 \\ -.20 & .90 & -.20 & -.30 & 0 \\ -.30 & -.35 & .50 & -.20 & 0 \\ -.30 & -.20 & -.20 & .70 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -.799 & 0 \\ 0 & 1 & 0 & -.836 & 0 \\ 0 & 0 & 1 & -1.465 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solve for the basic variables in terms of the free variable p_T, and obtain $p_A = .799p_T$, $p_M = .836p_T$, and $p_S = 1.465p_T$. Take $p_T = \$10.00$ and round off the other prices to $p_A = \$7.99$, $p_M = \$8.36$, and $p_S = \$14.65$ per unit.

c. Construct the new exchange table one column at a time. The entries in each column must sum to 1.

	Distribution of Output From:					Purchased by:
	Agriculture	Manufacturing	Services	Transportation		
output	↓	↓	↓	↓	input	
	.20	.35	.10	.20	→	Agriculture
	.10	.10	.20	.30	→	Manufacturing
	.40	.35	.50	.20	→	Services
	.30	.20	.20	.30	→	Transportation

d. [M] The new equilibrium equations are

$$\begin{aligned} p_A &= .20p_A + .35p_M + .10p_S + .20p_T \\ p_M &= .10p_A + .10p_M + .20p_S + .30p_T \\ p_S &= .40p_A + .35p_M + .50p_S + .20p_T \\ p_T &= .30p_A + .20p_M + .20p_S + .30p_T \end{aligned}$$

Move all variables to the left side and combine like terms:

$$\begin{aligned} .80p_A - .35p_M - .10p_S - .20p_T &= 0 \\ -.10p_A + .90p_M - .20p_S - .30p_T &= 0 \\ -.40p_A - .35p_M + .50p_S - .20p_T &= 0 \\ -.30p_A - .20p_M - .20p_S + .70p_T &= 0 \end{aligned}$$

Reduce the augmented matrix to reduced echelon form:

$$\begin{bmatrix} .80 & -.35 & -.10 & -.20 & 0 \\ -.10 & .90 & -.20 & -.30 & 0 \\ -.40 & -.35 & .50 & -.20 & 0 \\ -.30 & -.20 & -.20 & .70 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & -.781 & 0 \\ 0 & \textcircled{1} & 0 & -.767 & 0 \\ 0 & 0 & \textcircled{1} & -1.562 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solve for the basic variables in terms of the free variable p_T, and obtain $p_A = .781p_T$, $p_M = .767p_T$, and

$p_S = 1.562p_T$. Take $p_T = \$10.00$ and round off the other prices to $p_A = \$7.81$, $p_M = \$7.67$, and $p_S = \$15.62$ per unit. The campaign has caused unit prices for the Agriculture and Manufacturing sectirs to go down slightly, while increasing the unit price for the Services sector to increase by \$.10 per unit. The campaign has benefited the Services sector the most.

6. The following vectors list the numbers of atoms of aluminum (Al), oxygen (O), and carbon (C):

$$Al_2O_3: \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \quad C: \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad Al: \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad CO_2: \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \begin{matrix} \text{aluminum} \\ \text{oxygen} \\ \text{carbon} \end{matrix}$$

The coefficients in the equation $x_1 \cdot Al_2O_3 + x_2 \cdot C \rightarrow x_3 \cdot Al + x_4 \cdot CO_2$ satisfy

$$x_1\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x_3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Move the right terms to the left side (changing the sign of each entry in the third and fourth vectors) and row reduce the augmented matrix of the homogeneous system:

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 & 0 & 0 \\ 3 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 3/2 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 3/2 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -4/3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2/3 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -4/3 & 0 \end{bmatrix}$$

The general solution is $x_1 = (2/3)x_4$, $x_2 = x_4$, $x_3 = (4/3)x_4$, with x_4 free. Take $x_4 = 3$. Then $x_1 = 2$, $x_2 = 3$, and $x_3 = 4$. The balanced equation is

$$2Al_2O_3 + 3C \rightarrow 4Al + 3CO_2$$

7. The following vectors list the numbers of atoms of sodium (Na), hydrogen (H), carbon (C), and oxygen (O):

$$NaHCO_3: \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \quad H_3C_6H_5O_7: \begin{bmatrix} 0 \\ 8 \\ 6 \\ 7 \end{bmatrix}, \quad Na_3C_6H_5O_7: \begin{bmatrix} 3 \\ 5 \\ 6 \\ 7 \end{bmatrix}, \quad H_2O: \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad CO_2: \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \begin{matrix} \text{sodium} \\ \text{hydrogen} \\ \text{carbon} \\ \text{oxygen} \end{matrix}$$

The order of the various atoms is not important. The list here was selected by writing the elements in the order in which they first appear in the chemical equation, reading left to right:

$$x_1 \cdot NaHCO_3 + x_2 \cdot H_3C_6H_5O_7 \rightarrow x_3 \cdot Na_3C_6H_5O_7 + x_4 \cdot H_2O + x_5 \cdot CO_2.$$

The coefficients $x_1, \ldots, x_5$ satisfy the vector equation

$$x_1\begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 8 \\ 6 \\ 7 \end{bmatrix} = x_3\begin{bmatrix} 3 \\ 5 \\ 6 \\ 7 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} + x_5\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

Move all the terms to the left side (changing the sign of each entry in the third, fourth, and fifth vectors) and reduce the augmented matrix:

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 0 \\ 1 & 8 & -5 & -2 & 0 & 0 \\ 1 & 6 & -6 & 0 & -1 & 0 \\ 3 & 7 & -7 & -1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 0 \\ 0 & 0 & 1 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

The general solution is $x_1 = x_5$, $x_2 = (1/3)x_5$, $x_3 = (1/3)x_5$, $x_4 = x_5$, and x_5 is free. Take $x_5 = 3$. Then $x_1 = x_4 = 3$, and $x_2 = x_3 = 1$. The balanced equation is

$$3NaHCO_3 + H_3C_6H_5O_7 \rightarrow Na_3C_6H_5O_7 + 3H_2O + 3CO_2$$

8. The following vectors list the numbers of atoms of hydrogen (H), oxygen (O), calcium (Ca), and carbon (C):

$$H_3O:\begin{bmatrix}3\\1\\0\\0\end{bmatrix},\ CaCO_3:\begin{bmatrix}0\\3\\1\\1\end{bmatrix},\ H_2O:\begin{bmatrix}2\\1\\0\\0\end{bmatrix},\ Ca:\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\ CO_2:\begin{bmatrix}0\\2\\0\\1\end{bmatrix}\begin{matrix}\text{hydrogen}\\\text{oxygen}\\\text{calcium}\\\text{carbon}\end{matrix}$$

The coefficients in the chemical equation

$$x_1{\cdot}H_3O + x_2{\cdot}CaCO_3 \ \rightarrow\ x_3{\cdot}H_2O + x_4{\cdot}Ca + x_5{\cdot}CO_2$$

satisfy the vector equation

$$x_1\begin{bmatrix}3\\1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\3\\1\\1\end{bmatrix}=x_3\begin{bmatrix}2\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}0\\0\\1\\0\end{bmatrix}+x_5\begin{bmatrix}0\\2\\0\\1\end{bmatrix}$$

Move the terms to the left side (changing the sign of each entry in the last three vectors) and reduce the augmented matrix:

$$\begin{bmatrix}3&0&-2&0&0&0\\1&3&-1&0&-2&0\\0&1&0&-1&0&0\\0&1&0&0&-1&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&0&-2&0\\0&1&0&0&-1&0\\0&0&1&0&-3&0\\0&0&0&1&-1&0\end{bmatrix}$$

The general solution is $x_1 = 2x_5$, $x_2 = x_5$, $x_3 = 3x_5$, $x_4 = x_5$, and x_5 is free. Take $x_5 = 1$. Then $x_1 = 2$, and $x_2 = x_4 = 1$, and $x_3 = 3$. The balanced equation is

$$2H_3O + CaCO_3 \ \rightarrow\ 3H_2O + Ca + CO_2$$

9. The following vectors list the numbers of atoms of boron (B), sulfur (S), hydrogen (H), and oxygen (O):

$$B_2S_3:\begin{bmatrix}2\\3\\0\\0\end{bmatrix},\ H_2O:\begin{bmatrix}0\\0\\2\\1\end{bmatrix},\ H_3BO_3:\begin{bmatrix}1\\0\\3\\3\end{bmatrix},\ H_2S:\begin{bmatrix}0\\1\\2\\0\end{bmatrix}\begin{matrix}\text{boron}\\\text{sulfur}\\\text{hydrogen}\\\text{oxygen}\end{matrix}$$

The coefficients in the equation $x_1{\cdot}B_2S_3 + x_2{\cdot}H_2O \rightarrow x_3{\cdot}H_3BO_3 + x_4{\cdot}H_2S$ satisfy

$$x_1\begin{bmatrix}2\\3\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\0\\2\\1\end{bmatrix}=x_3\begin{bmatrix}1\\0\\3\\3\end{bmatrix}+x_4\begin{bmatrix}0\\1\\2\\0\end{bmatrix}$$

Move the terms to the left side (changing the sign of each entry in the third and fourth vectors) and row reduce the augmented matrix of the homogeneous system:

$$\begin{bmatrix}2&0&-1&0&0\\3&0&0&-1&0\\0&2&-3&-2&0\\0&1&-3&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&-1/3&0\\0&1&0&-2&0\\0&0&1&-2/3&0\\0&0&0&0&0\end{bmatrix}$$

The general solution is $x_1 = (1/3)\,x_4$, $x_2 = 2x_4$, $x_3 = (2/3)\,x_4$, with x_4 free. Take $x_4 = 3$. Then $x_1 = 1$, $x_2 = 6$, and $x_3 = 2$. The balanced equation is

$$B_2S_3 + 6H_2O \ \rightarrow\ 2H_3BO_3 + 3H_2S$$

10. **[M]** Set up vectors that list the atoms per molecule. Using the order lead (Pb), nitrogen (N), chromium (Cr), manganese (Mn), and oxygen (O), the vector equation to be solved is

$$x_1\begin{bmatrix}1\\6\\0\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\0\\1\\2\\8\end{bmatrix}=x_3\begin{bmatrix}3\\0\\0\\0\\4\end{bmatrix}+x_4\begin{bmatrix}0\\0\\2\\0\\3\end{bmatrix}+x_5\begin{bmatrix}0\\0\\0\\1\\2\end{bmatrix}+x_6\begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix}\begin{matrix}\text{lead}\\\text{nitrogen}\\\text{chromium}\\\text{manganese}\\\text{oxygen}\end{matrix}$$

The general solution is $x_1 = (1/6)x_6$, $x_2 = (22/45)x_6$, $x_3 = (1/18)x_6$, $x_4 = (11/45)x_6$, $x_5 = (44/45)x_6$, and x_6 is free. Take $x_6 = 90$. Then $x_1 = 15$, $x_2 = 44$, $x_3 = 5$, $x_4 = 22$, and $x_5 = 88$. The balanced equation is

$$15PbN_6 + 44CrMn_2O_8 \ \rightarrow\ 5Pb_3O_4 + 22Cr_2O_3 + 88MnO_2 + 90NO$$

11. **[M]** Set up vectors that list the atoms per molecule. Using the order manganese (Mn), sulfur (S), arsenic (As), chromium (Cr), oxygen (O), and hydrogen (H), the vector equation to be solved is

$$x_1\begin{bmatrix}1\\1\\0\\0\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\0\\2\\10\\35\\0\end{bmatrix}+x_3\begin{bmatrix}0\\1\\0\\0\\4\\2\end{bmatrix}=x_4\begin{bmatrix}1\\0\\0\\0\\4\\1\end{bmatrix}+x_5\begin{bmatrix}0\\0\\1\\0\\0\\3\end{bmatrix}+x_6\begin{bmatrix}0\\3\\0\\1\\12\\0\end{bmatrix}+x_7\begin{bmatrix}0\\0\\0\\0\\1\\2\end{bmatrix}\begin{matrix}\text{manganese}\\\text{sulfur}\\\text{arsenic}\\\text{chromium}\\\text{oxygen}\\\text{hydrogen}\end{matrix}$$

In rational format, the general solution is $x_1 = (16/327)x_7$, $x_2 = (13/327)x_7$, $x_3 = (374/327)x_7$, $x_4 = (16/327)x_7$, $x_5 = (26/327)x_7$, $x_6 = (130/327)x_7$, and x_7 is free. Take $x_7 = 327$ to make the other variables whole numbers. The balanced equation is

$$16MnS + 13As_2Cr_{10}O_{35} + 374H_2SO_4 \ \rightarrow\ 16HMnO_4 + 26AsH_3 + 130CrS_3O_{12} + 327H_2O$$

Note that some students may use decimal calculation and simply "round off" the fractions that relate $x_1, \ldots, x_6$ to x_7. The equations they construct may balance most of the elements but miss an atom or two. Here is a solution submitted by two of my students:

$$5MnS + 4As_2Cr_{10}O_{35} + 115H_2SO_4 \ \rightarrow\ 5HMnO_4 + 8AsH_3 + 40CrS_3O_{12} + 100H_2O$$

Everything balances except the hydrogen. The right side is short 1 hydrogen atom. Perhaps the students thought that it escaped!

12. Write the equations for each intersection:

Intersection	Flow in		Flow out
A	$x_1 + x_4$	=	x_2
B	x_2	=	$x_3 + 100$
C	$x_3 + 80$	=	x_4

Rearrange the equations:

$$\begin{aligned} x_1 - x_2 \quad\quad + x_4 &= 0 \\ x_2 - x_3 \quad\quad &= 100 \\ x_3 - x_4 &= -80 \end{aligned}$$

Reduce the augmented matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 1 & -1 & -80 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 0 & 20 \\ 0 & \textcircled{1} & 0 & -1 & 20 \\ 0 & 0 & \textcircled{1} & -1 & -80 \end{bmatrix}$$

The general solution (written in the style of Section 1.2) is

$$\begin{cases} x_1 = 20 \\ x_2 = 20 + x_4 \\ x_3 = -80 + x_4 \\ x_4 \text{ is free} \end{cases}$$

Since x_3 cannot be negative, the minimum value of x_4 is 80.

13. Write the equations for each intersection:

Intersection	Flow in		Flow out
A	$x_2 + 30$	=	$x_1 + 80$
B	$x_3 + x_5$	=	$x_2 + x_4$
C	$x_6 + 100$	=	$x_5 + 40$
D	$x_4 + 40$	=	$x_6 + 90$
E	$x_1 + 60$	=	$x_3 + 20$

Rearrange the equations:

$$\begin{aligned} x_1 - x_2 &= -50 \\ x_2 - x_3 + x_4 - x_5 &= 0 \\ x_5 - x_6 &= 60 \\ x_4 - x_6 &= 50 \\ x_1 - x_3 &= -40 \end{aligned}$$

Reduce the augmented matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 1 & 0 & -1 & 0 & 0 & 0 & -40 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \textcircled{1} & 0 & -1 & 0 & 0 & 0 & -40 \\ 0 & \textcircled{1} & -1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & \textcircled{1} & 0 & -1 & 50 \\ 0 & 0 & 0 & 0 & \textcircled{1} & -1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a. The general solution is $\begin{cases} x_1 = x_3 - 40 \\ x_2 = x_3 + 10 \\ x_3 \text{ is free} \\ x_4 = x_6 + 50 \\ x_5 = x_6 + 60 \\ x_6 \text{ is free} \end{cases}$

b. To find minimum flows, note that since x_1 cannot be negative, $x_3 \geq 40$. This implies that $x_2 \geq 50$. Also, since x_6 cannot be negative, $x_4 \geq 50$ and $x_5 \geq 60$. The minimum flows are $x_2 = 50$, $x_3 = 40$, $x_4 = 50$, $x_5 = 60$ (when $x_1 = 0$ and $x_6 = 0$).

14. Write the equations for each intersection:

Intersection	Flow in		Flow out
A	80	=	$x_1 + x_5$
B	$x_1 + x_2 + 100$	=	x_4
C	x_3	=	$x_2 + 90$
D	$x_4 + x_5$	=	$x_3 + 90$

Rearrange the equations:

$$\begin{aligned} x_1 + x_5 &= 80 \\ x_1 + x_2 - x_4 &= -100 \\ x_2 - x_3 &= -90 \\ x_3 - x_4 - x_5 &= -90 \end{aligned}$$

Reduce the augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 80 \\ 1 & 1 & 0 & -1 & 0 & -100 \\ 0 & 1 & -1 & 0 & 0 & -90 \\ 0 & 0 & 1 & -1 & -1 & -90 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 0 & 1 & 80 \\ 0 & \textcircled{1} & 0 & -1 & -1 & -180 \\ 0 & 0 & \textcircled{1} & -1 & -1 & -90 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a. The general solution is $\begin{cases} x_1 = 80 - x_5 \\ x_2 = x_4 + x_5 - 180 \\ x_3 = x_4 + x_5 - 90 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \end{cases}$

b. If $x_5 = 0$, then the general solution is $\begin{cases} x_1 = 80 \\ x_2 = x_4 - 180 \\ x_3 = x_4 - 90 \\ x_4 \text{ is free} \end{cases}$

c. Since x_2 cannot be negative, the minimum value of x_4 when $x_5 = 0$ is 180.

15. Write the equations for each intersection.

Intersection	Flow in		Flow out
A	$x_6 + 60$	=	x_1
B	x_1	=	$x_2 + 70$
C	$x_2 + 100$	=	x_3
D	x_3	=	$x_4 + 90$
E	$x_4 + 80$	=	x_5
F	x_5	=	$x_6 + 80$

Rearrange the equations:

$$\begin{aligned}
x_1 \qquad\qquad\qquad\qquad\qquad\qquad - x_6 &= 60\\
x_1 - x_2 \qquad\qquad\qquad\qquad\qquad &= 70\\
x_2 - x_3 \qquad\qquad\qquad\qquad &= -100\\
x_3 - x_4 \qquad\qquad\qquad &= 90\\
x_4 - x_5 \qquad\qquad &= -80\\
x_5 - x_6 &= 80
\end{aligned}$$

Reduce the augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 60 \\ 1 & -1 & 0 & 0 & 0 & 0 & 70 \\ 0 & 1 & -1 & 0 & 0 & 0 & -100 \\ 0 & 0 & 1 & -1 & 0 & 0 & 90 \\ 0 & 0 & 0 & 1 & -1 & 0 & -80 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 60 \\ 0 & 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 0 & 1 & 0 & 0 & -1 & 90 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is $\begin{cases} x_1 = 60 + x_6 \\ x_2 = -10 + x_6 \\ x_3 = 90 + x_6 \\ x_4 = x_6 \\ x_5 = 80 + x_6 \\ x_6 \text{ is free} \end{cases}$.

Since x_2 cannot be negative, the minimum value of x_6 is 10.

Note: The MATLAB box in the *Study Guide* discusses rational calculations, needed for balancing the chemical equations in Exercises 10 and 11. As usual, the appendices cover this material for Maple, Mathematica, and the TI calculators.