Math 260 Homework 1.4

1.4 SOLUTIONS

Notes: Key exercises are 1–20, 27, 28, 31 and 32. Exercises 29, 30, 33, and 34 are harder. Exercise 34 anticipates the Invertible Matrix Theorem but is not used in the proof of that theorem.

1. The matrix-vector $A\mathbf{x}$ is not defined because the number of columns (2) in the 3×2 matrix A does not match the number of entries (3) in the vector **x**.

2. The matrix-vector $A\mathbf{x}$ is not defined because the number of columns (1) in the 3×1 matrix A does not match the number of entries (2) in the vector **x**.

3. $A\mathbf{x}=\begin{bmatrix}1&2\\-3&1\\1&6\end{bmatrix}\begin{bmatrix}-2\\3\end{bmatrix}=(-2)\begin{bmatrix}1\\-3\\1\end{bmatrix}+3\begin{bmatrix}2\\1\\6\end{bmatrix}=\begin{bmatrix}-2\\6\\-2\end{bmatrix}+\begin{bmatrix}6\\3\\18\end{bmatrix}=\begin{bmatrix}4\\9\\16\end{bmatrix}$, and

$$A\mathbf{x}=\begin{bmatrix}1&2\\-3&1\\1&6\end{bmatrix}\begin{bmatrix}-2\\3\end{bmatrix}=\begin{bmatrix}1\cdot(-2)+2\cdot3\\(-3)\cdot(-2)+1\cdot3\\1\cdot(-2)+6\cdot3\end{bmatrix}=\begin{bmatrix}4\\9\\16\end{bmatrix}$$

4. $A\mathbf{x}=\begin{bmatrix}1&3&-4\\3&2&1\end{bmatrix}\begin{bmatrix}1\\2\\1\end{bmatrix}=1\cdot\begin{bmatrix}1\\3\end{bmatrix}+2\cdot\begin{bmatrix}3\\2\end{bmatrix}+1\cdot\begin{bmatrix}-4\\1\end{bmatrix}=\begin{bmatrix}1+6-4\\3+4+1\end{bmatrix}=\begin{bmatrix}3\\8\end{bmatrix}$, and

$$A\mathbf{x}=\begin{bmatrix}1&3&-4\\3&2&1\end{bmatrix}\begin{bmatrix}1\\2\\1\end{bmatrix}=\begin{bmatrix}1\cdot1+3\cdot2+(-4)\cdot1\\3\cdot1+2\cdot2+1\cdot1\end{bmatrix}=\begin{bmatrix}3\\8\end{bmatrix}$$

5. On the left side of the matrix equation, use the entries in the vector **x** as the weights in a linear combination of the columns of the matrix A:

$$2\cdot\begin{bmatrix}1\\-2\end{bmatrix}-1\cdot\begin{bmatrix}2\\-3\end{bmatrix}+1\cdot\begin{bmatrix}-3\\1\end{bmatrix}-1\cdot\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-4\\1\end{bmatrix}$$

6. On the left side of the matrix equation, use the entries in the vector **x** as the weights in a linear combination of the columns of the matrix A:

$$-3\cdot\begin{bmatrix}2\\3\\8\\-2\end{bmatrix}+5\cdot\begin{bmatrix}-3\\2\\-5\\1\end{bmatrix}=\begin{bmatrix}-21\\1\\-49\\11\end{bmatrix}$$

7. The left side of the equation is a linear combination of three vectors. Write the matrix A whose columns are those three vectors, and create a variable vector **x** with three entries:

$$A=\left[\begin{bmatrix}4\\-1\\7\\-4\end{bmatrix}\ \begin{bmatrix}-5\\3\\-5\\1\end{bmatrix}\ \begin{bmatrix}7\\-8\\0\\2\end{bmatrix}\right]=\begin{bmatrix}4&-5&7\\-1&3&-8\\7&-5&0\\-4&1&2\end{bmatrix}$$ and $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$. Thus the equation $A\mathbf{x}=\mathbf{b}$ is

$$\begin{bmatrix}4&-5&7\\-1&3&-8\\7&-5&0\\-4&1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}6\\-8\\0\\-7\end{bmatrix}$$

For your information: The unique solution of this equation is (5, 7, 3). Finding the solution by hand would be time-consuming.

Note: The skill of writing a vector equation as a matrix equation will be important for both theory and application throughout the text. See also Exercises 27 and 28.

8. The left side of the equation is a linear combination of four vectors. Write the matrix A whose columns are those four vectors, and create a variable vector with four entries:

$$A=\left[\begin{bmatrix}2\\-4\end{bmatrix}\ \begin{bmatrix}-1\\5\end{bmatrix}\ \begin{bmatrix}-4\\3\end{bmatrix}\ \begin{bmatrix}0\\2\end{bmatrix}\right]=\begin{bmatrix}2&-1&-4&0\\-4&5&3&2\end{bmatrix}$$, and $\mathbf{z}=\begin{bmatrix}z_1\\z_2\\z_3\\z_4\end{bmatrix}$. Then the equation $A\mathbf{z}=\mathbf{b}$

is $\begin{bmatrix}2&-1&-4&0\\-4&5&3&2\end{bmatrix}\begin{bmatrix}z_1\\z_2\\z_3\\z_4\end{bmatrix}=\begin{bmatrix}5\\12\end{bmatrix}$.

For your information: One solution is (8, 7, 1, 3). The general solution is $z_1 = 37/6 + (17/6)z_3 - (1/3)z_4$, $z_2 = 22/3 + (5/3)z_3 - (2/3)z_4$, with z_3 and z_4 free.

9. The system has the same solution set as the vector equation

$$x_1\begin{bmatrix}5\\0\end{bmatrix}+x_2\begin{bmatrix}1\\2\end{bmatrix}+x_3\begin{bmatrix}-3\\4\end{bmatrix}=\begin{bmatrix}8\\0\end{bmatrix}$$

and this equation has the same solution set as the matrix equation

$$\begin{bmatrix}5&1&-3\\0&2&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}8\\0\end{bmatrix}$$

10. The system has the same solution set as the vector equation

$$x_1\begin{bmatrix}4\\5\\3\end{bmatrix}+x_2\begin{bmatrix}-1\\3\\-1\end{bmatrix}=\begin{bmatrix}8\\2\\1\end{bmatrix}$$

and this equation has the same solution set as the matrix equation

$$\begin{bmatrix} 4 & -1 \\ 5 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 1 \end{bmatrix}$$

11. To solve $A\mathbf{x} = \mathbf{b}$, row reduce the augmented matrix $[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3\ \ \mathbf{b}]$ for the corresponding linear system:

$$\begin{bmatrix} 1 & 3 & -4 & -2 \\ 1 & 5 & 2 & 4 \\ -3 & -7 & 6 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -4 & -2 \\ 0 & 2 & 6 & 6 \\ 0 & 2 & -6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -4 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The solution is $\begin{cases} x_1 = -11 \\ x_2 = 3 \\ x_3 = 0 \end{cases}$. As a vector, the solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11 \\ 3 \\ 0 \end{bmatrix}$.

12. To solve $A\mathbf{x} = \mathbf{b}$, row reduce the augmented matrix $[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3\ \ \mathbf{b}]$ for the corresponding linear system:

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ -3 & -4 & 2 & 2 \\ 5 & 2 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & -8 & 8 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -8 & 8 & -8 \\ 0 & 2 & -1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -1 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The solution is $\begin{cases} x_1 = -4 \\ x_2 = 4 \\ x_3 = 3 \end{cases}$. As a vector, the solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 3 \end{bmatrix}$.

13. The vector $\mathbf{u}$ is in the plane spanned by the columns of A if and only if $\mathbf{u}$ is a linear combination of the columns of A. This happens if and only if the equation $A\mathbf{x} = \mathbf{u}$ has a solution. (See the box preceding Example 3 in Section 1.4.) To study this equation, reduce the augmented matrix $[A\ \ \mathbf{u}]$

$$\begin{bmatrix} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ -2 & 6 & 4 \\ 3 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 8 & 12 \\ 0 & -8 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 8 & 12 \\ 0 & 0 & 0 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{u}$ has a solution, so $\mathbf{u}$ is in the plane spanned by the columns of A.

For your information: The unique solution of $A\mathbf{x} = \mathbf{u}$ is (5/2, 3/2).

14. Reduce the augmented matrix $[A\ \ \mathbf{u}]$ to echelon form:

$$\begin{bmatrix} 2 & 5 & -1 & 4 \\ 0 & 1 & -1 & -1 \\ 1 & 2 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & -1 \\ 2 & 5 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{u}$ has no solution, so $\mathbf{u}$ is not in the subset spanned by the columns of A.

15. The augmented matrix for $A\mathbf{x} = \mathbf{b}$ is $\begin{bmatrix} 3 & -1 & b_1 \\ -9 & 3 & b_2 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 3 & -1 & b_1 \\ 0 & 0 & b_2 + 3b_1 \end{bmatrix}$.

This shows that the equation $A\mathbf{x} = \mathbf{b}$ is not consistent when $3b_1 + b_2$ is nonzero. The set of $\mathbf{b}$ for which the equation *is* consistent is a line through the origin–the set of all points (b_1, b_2) satisfying $b_2 = -3b_1$.

16. Row reduce the augmented matrix $[A\ \ \mathbf{b}]$: $A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ -2 & 2 & 0 & b_2 \\ 4 & -1 & 3 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -2 & -2 & b_2 + 2b_1 \\ 0 & 7 & 7 & b_3 - 4b_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -2 & -2 & b_2 + 2b_1 \\ 0 & 0 & 0 & b_3 - 4b_1 + (7/2)(b_2 + 2b_1) \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -2 & -2 & b_2 + 2b_1 \\ 0 & 0 & 0 & 3b_1 + (7/2)b_2 + b_3 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $3b_1 + (7/2)b_2 + b_3 = 0$, or $6b_1 + 7b_2 + 2b_3 = 0$. The set of such $\mathbf{b}$ is a plane through the origin in $\mathbf{R}^3$.

17. Row reduction shows that only three rows of A contain a pivot position:

$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -4 & 2 & -8 \\ 0 & -6 & 3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Because not every row of A contains a pivot position, Theorem 4 in Section 1.4 shows that the equation $A\mathbf{x} = \mathbf{b}$ does *not* have a solution for each $\mathbf{b}$ in $\mathbf{R}^4$.

18. Row reduction shows that only three rows of B contain a pivot position:

$$B = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 0 & 1 & 3 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Because not every row of B contains a pivot position, Theorem 4 in Section 1.4 shows that not all vectors in $\mathbf{R}^4$ can be written as a linear combination of the columns of B. The columns of B certainly do *not* span $\mathbf{R}^3$, because each column of B is in $\mathbf{R}^4$, not $\mathbf{R}^3$. (This question was asked to alert students to a fairly common misconception among students who are just learning about spanning.)

19. The work in Exercise 17 shows that statement (d) in Theorem 4 is false. So all four statements in Theorem 4 are false. Thus, not all vectors in $\mathbf{R}^4$ can be written as a linear combination of the columns of A. Also, the columns of A do *not* span $\mathbf{R}^4$.

20. The work in Exercise 18 shows that statement (d) in Theorem 4 is false. So all four statements in Theorem 4 are false. Thus, the equation $B\mathbf{x} = \mathbf{y}$ does *not* have a solution for each $\mathbf{y}$ in $\mathbf{R}^4$, and the columns of B do *not* span $\mathbf{R}^4$.

21. Row reduce the matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3]$ to determine whether it has a pivot in each row.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 1 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3]$ does not have a pivot in each row, so the columns of the matrix do not span $\mathbf{R}^4$, by Theorem 4. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ does not span $\mathbf{R}^4$.

Note: Some students may realize that row operations are not needed, and thereby discover the principle covered in Exercises 31 and 32.

22. Row reduce the matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3]$ to determine whether it has a pivot in each row.

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -2 \\ -3 & 9 & -6 \end{bmatrix} \sim \begin{bmatrix} \textcircled{-3} & 9 & -6 \\ 0 & \textcircled{-3} & -2 \\ 0 & 0 & \textcircled{4} \end{bmatrix}$$

The matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3]$ has a pivot in each row, so the columns of the matrix span $\mathbf{R}^3$, by Theorem 4. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbf{R}^3$.

23. **a.** False. See the paragraph following equation (3). The text calls $A\mathbf{x} = \mathbf{b}$ a *matrix equation*.

b. True. See the box before Example 3.

c. False. See the warning following Theorem 4.

d. True. See Example 4.

e. True. See parts (c) and (a) in Theorem 4.

f. True. In Theorem 4, statement (a) is false if and only if statement (d) is also false.

24. **a.** True. This statement is in Theorem 3. However, the statement is true without any "proof" because, by definition, $A\mathbf{x}$ is simply a notation for $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$, where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are the columns of A.

b. True. See the box before Example 3.

c. True. See Example 2.

d. False. In Theorem 4, statement (d) is true if and only if statement (a) is true.

e. True. See Theorem 3.

f. False. In Theorem 4, statement (c) is false if and only if statement (a) is also false.

25. By definition, the matrix-vector product on the left is a linear combination of the columns of the matrix, in this case using weights –3, –1, and 2. So $c_1 = -3$, $c_2 = -1$, and $c_3 = 2$.

26. The equation in x_1 and x_2 involves the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, and it may be viewed as

$[\mathbf{u} \;\; \mathbf{v}]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{w}$. By definition of a matrix-vector product, $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{w}$. The stated fact that $2\mathbf{u} - 3\mathbf{v} - \mathbf{w} = \mathbf{0}$ can be rewritten as $2\mathbf{u} - 3\mathbf{v} = \mathbf{w}$. So, a solution is $x_1 = 2$, $x_2 = -3$.

27. The matrix equation can be written as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = \mathbf{v}_6$, where $c_1 = -3$, $c_2 = 1$, $c_3 = 2$, $c_4 = -1$, $c_5 = 2$, and

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 9 \\ -2 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 7 \\ -4 \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} 11 \\ -11 \end{bmatrix}$$

28. Place the vectors $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ into the columns of a matrix, say, Q and place the weights x_1, x_2, and x_3 into a vector, say, $\mathbf{x}$. Then the vector equation becomes

$$Q\mathbf{x} = \mathbf{v}, \text{ where } Q = [\mathbf{q}_1 \;\; \mathbf{q}_2 \;\; \mathbf{q}_3] \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Note: If your answer is the equation $A\mathbf{x} = \mathbf{b}$, you need to specify what A and $\mathbf{b}$ are.

29. Start with any 3×3 matrix B in echelon form that has three pivot positions. Perform a row operation (a row interchange or a row replacement) that creates a matrix A that is *not* in echelon form. Then A has the desired property. The justification is given by row reducing A to B, in order to display the pivot positions. Since A has a pivot position in every row, the columns of A span $\mathbf{R}^3$, by Theorem 4.

30. Start with any nonzero 3×3 matrix B in echelon form that has fewer than three pivot positions. Perform a row operation that creates a matrix A that is *not* in echelon form. Then A has the desired property. Since A does not have a pivot position in every row, the columns of A do not span $\mathbf{R}^3$, by Theorem 4.

31. A 3×2 matrix has three rows and two columns. With only two columns, A can have at most two pivot columns, and so A has at most two pivot positions, which is not enough to fill all three rows. By Theorem 4, the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbf{R}^3$. Generally, if A is an $m \times n$ matrix with $m > n$, then A can have at most n pivot positions, which is not enough to fill all m rows. Thus, the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbf{R}^3$.

32. A set of three vectors in $\mathbf{R}^4$ cannot span $\mathbf{R}^4$. Reason: the matrix A whose columns are these three vectors has four rows. To have a pivot in each row, A would have to have at least four columns (one for each pivot), which is not the case. Since A does not have a pivot in every row, its columns do not span $\mathbf{R}^4$, by Theorem 4. In general, a set of n vectors in $\mathbf{R}^m$ cannot span $\mathbf{R}^m$ when n is less than m.

33. If the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the associated system of equations does not have any free variables. If every variable is a basic variable, then each column of A is a pivot column. So

the reduced echelon form of A must be $\begin{bmatrix} \textcircled{1} & 0 & 0 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 \end{bmatrix}$.

Note: Exercises 33 and 36 are difficult in the context of this section because the focus in Section 1.4 is on existence of solutions, not uniqueness. However, these exercises serve to review ideas from Section 1.2, and they anticipate ideas that will come later.

34. Given $A\mathbf{u}_1 = \mathbf{v}_1$ and $A\mathbf{u}_2 = \mathbf{v}_2$, you are asked to show that the equation $A\mathbf{x} = \mathbf{w}$ has a solution, where $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$. Observe that $\mathbf{w} = A\mathbf{u}_1 + A\mathbf{u}_2$ and use Theorem 5(a) with $\mathbf{u}_1$ and $\mathbf{u}_2$ in place of $\mathbf{u}$ and $\mathbf{v}$, respectively. That is, $\mathbf{w} = A\mathbf{u}_1 + A\mathbf{u}_2 = A(\mathbf{u}_1 + \mathbf{u}_2)$. So the vector $\mathbf{x} = \mathbf{u}_1 + \mathbf{u}_2$ is a solution of $\mathbf{w} = A\mathbf{x}$.