Math 260 Homework 1.3

1. $\mathbf{u}+\mathbf{v}=\begin{bmatrix}-1\\2\end{bmatrix}+\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1+(-3)\\2+(-1)\end{bmatrix}=\begin{bmatrix}-4\\1\end{bmatrix}$.

Using the definitions carefully,

$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}-1\\2\end{bmatrix}+(-2)\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}+\begin{bmatrix}(-2)(-3)\\(-2)(-1)\end{bmatrix}=\begin{bmatrix}-1+6\\2+2\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix}$$, or, more quickly,

$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}-1\\2\end{bmatrix}-2\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1+6\\2+2\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix}$$. The intermediate step is often not written.

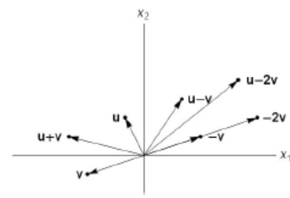
2. $\mathbf{u}+\mathbf{v}=\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}3+2\\2-1\end{bmatrix}=\begin{bmatrix}5\\1\end{bmatrix}$.

Using the definitions carefully,

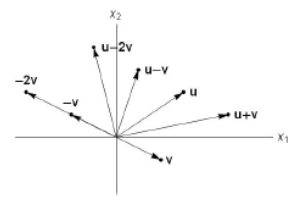
$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}3\\2\end{bmatrix}+(-2)\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}(-2)(2)\\(-2)(-1)\end{bmatrix}=\begin{bmatrix}3-4\\2+2\end{bmatrix}=\begin{bmatrix}-1\\4\end{bmatrix}$$, or, more quickly,

$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}3\\2\end{bmatrix}-2\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}3-4\\2+2\end{bmatrix}=\begin{bmatrix}-1\\4\end{bmatrix}$$. The intermediate step is often not written.

3.



4.



5. $x_1\begin{bmatrix}3\\-2\\8\end{bmatrix}+x_2\begin{bmatrix}5\\0\\-9\end{bmatrix}=\begin{bmatrix}2\\-3\\8\end{bmatrix}$, $\begin{bmatrix}3x_1\\-2x_1\\8x_1\end{bmatrix}+\begin{bmatrix}5x_2\\0\\-9x_2\end{bmatrix}=\begin{bmatrix}2\\-3\\8\end{bmatrix}$, $\begin{bmatrix}3x_1+5x_2\\-2x_1\\8x_1-9x_2\end{bmatrix}=\begin{bmatrix}2\\-3\\8\end{bmatrix}$

$$\begin{aligned}3x_1 + 5x_2 &= 2\\ -2x_1 \qquad &= -3\\ 8x_1 - 9x_2 &= 8\end{aligned}$$

Usually the intermediate steps are not displayed.

6. $x_1\begin{bmatrix}3\\-2\end{bmatrix}+x_2\begin{bmatrix}7\\3\end{bmatrix}+x_3\begin{bmatrix}-2\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}3x_1\\-2x_1\end{bmatrix}+\begin{bmatrix}7x_2\\3x_2\end{bmatrix}+\begin{bmatrix}-2x_3\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}3x_1+7x_2-2x_3\\-2x_1+3x_2+x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$$\begin{aligned}3x_2 + 7x_2 - 2x_3 &= 0\\ -2x_1 + 3x_2 + x_3 &= 0\end{aligned}$$

Usually the intermediate steps are not displayed.

7. See the figure below. Since the grid can be extended in every direction, the figure suggests that every vector in $\mathbf{R}^2$ can be written as a linear combination of **u** and **v**.

To write a vector **a** as a linear combination of **u** and **v**, imagine walking from the origin to **a** along the grid "streets" and keep track of how many "blocks" you travel in the **u**-direction and how many in the **v**-direction.

a. To reach **a** from the origin, you might travel 1 unit in the **u**-direction and –2 units in the **v**-direction (that is, 2 units in the negative **v**-direction). Hence **a** = **u** – 2**v**.

b. To reach **b** from the origin, travel 2 units in the **u**-direction and –2 units in the **v**-direction. So **b** = 2**u** – 2**v**. Or, use the fact that **b** is 1 unit in the **u**-direction from **a**, so that

$\mathbf{b} = \mathbf{a} + \mathbf{u} = (\mathbf{u} - 2\mathbf{v}) + \mathbf{u} = 2\mathbf{u} - 2\mathbf{v}$

c. The vector **c** is –1.5 units from **b** in the **v**-direction, so

$\mathbf{c} = \mathbf{b} - 1.5\mathbf{v} = (2\mathbf{u} - 2\mathbf{v}) - 1.5\mathbf{v} = 2\mathbf{u} - 3.5\mathbf{v}$

d. The "map" suggests that you can reach **d** if you travel 3 units in the **u**-direction and –4 units in the **v**-direction. If you prefer to stay on the paths displayed on the map, you might travel from the origin to –3**v**, then move 3 units in the **u**-direction, and finally move –1 unit in the **v**-direction. So

$\mathbf{d} = -3\mathbf{v} + 3\mathbf{u} - \mathbf{v} = 3\mathbf{u} - 4\mathbf{v}$

Another solution is

$\mathbf{d} = \mathbf{b} - 2\mathbf{v} + \mathbf{u} = (2\mathbf{u} - 2\mathbf{v}) - 2\mathbf{v} + \mathbf{u} = 3\mathbf{u} - 4\mathbf{v}$

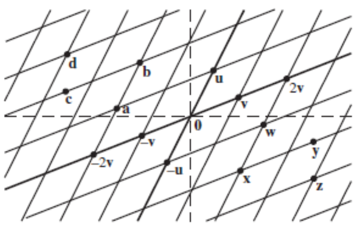


Figure for Exercises 7 and 8

8. See the figure above. Since the grid can be extended in every direction, the figure suggests that every vector in $\mathbf{R}^2$ can be written as a linear combination of **u** and **v**.

w. To reach **w** from the origin, travel –1 units in the **u**-direction (that is, 1 unit in the negative **u**-direction) and travel 2 units in the **v**-direction. Thus, $\mathbf{w} = (-1)\mathbf{u} + 2\mathbf{v}$, or $\mathbf{w} = 2\mathbf{v} - \mathbf{u}$.

x. To reach **x** from the origin, travel 2 units in the **v**-direction and –2 units in the **u**-direction. Thus, $\mathbf{x} = -2\mathbf{u} + 2\mathbf{v}$. Or, use the fact that **x** is –1 units in the **u**-direction from **w**, so that

$\mathbf{x} = \mathbf{w} - \mathbf{u} = (-\mathbf{u} + 2\mathbf{v}) - \mathbf{u} = -2\mathbf{u} + 2\mathbf{v}$

y. The vector **y** is 1.5 units from **x** in the **v**-direction, so

$\mathbf{y} = \mathbf{x} + 1.5\mathbf{v} = (-2\mathbf{u} + 2\mathbf{v}) + 1.5\mathbf{v} = -2\mathbf{u} + 3.5\mathbf{v}$

z. The map suggests that you can reach **z** if you travel 4 units in the **v**-direction and –3 units in the **u**-direction. So $\mathbf{z} = 4\mathbf{v} - 3\mathbf{u} = -3\mathbf{u} + 4\mathbf{v}$. If you prefer to stay on the paths displayed on the "map," you might travel from the origin to –2**u**, then 4 units in the **v**-direction, and finally move –1 unit in the **u**-direction. So

$\mathbf{z} = -2\mathbf{u} + 4\mathbf{v} - \mathbf{u} = -3\mathbf{u} + 4\mathbf{v}$

9.
$$\begin{aligned} x_2 + 5x_3 &= 0 \\ 4x_1 + 6x_2 - x_3 &= 0, \\ -x_1 + 3x_2 - 8x_3 &= 0 \end{aligned} \qquad \begin{bmatrix} x_2 + 5x_3 \\ 4x_1 + 6x_2 - x_3 \\ -x_1 + 3x_2 - 8x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 4x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 6x_2 \\ 3x_2 \end{bmatrix} + \begin{bmatrix} 5x_3 \\ -x_3 \\ -8x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad x_1\begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Usually, the intermediate calculations are not displayed.

10.
$$\begin{aligned} 3x_1 - 2x_2 + 4x_3 &= 3 \\ -2x_1 - 7x_2 + 5x_3 &= 1, \\ 5x_1 + 4x_2 - 3x_3 &= 2 \end{aligned} \qquad \begin{bmatrix} 3x_1 - 2x_2 + 4x_3 \\ -2x_1 - 7x_2 + 5x_3 \\ 5x_1 + 4x_2 - 3x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3x_1 \\ -2x_1 \\ 5x_1 \end{bmatrix} + \begin{bmatrix} -2x_2 \\ -7x_2 \\ 4x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ 5x_3 \\ -3x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \qquad x_1\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ -7 \\ 4 \end{bmatrix} + x_3\begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

Usually, the intermediate calculations are not displayed.

11. The question

Is **b** a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$?

is equivalent to the question

Does the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$ have a solution?

The equation

$$x_1\underset{\substack{\uparrow \\ \mathbf{a}_1}}{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}} + x_2\underset{\substack{\uparrow \\ \mathbf{a}_2}}{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}} + x_3\underset{\substack{\uparrow \\ \mathbf{a}_3}}{\begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}} = \underset{\substack{\uparrow \\ \mathbf{b}}}{\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}}$$

has the same solution set as the linear system whose augmented matrix is

$$M = \begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix}$$

Row reduce M until the pivot positions are visible:

$$M \sim \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 5 & 2 \\ 0 & \textcircled{1} & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The linear system corresponding to M *has* a solution, so the vector equation (*) has a solution, and therefore **b** *is* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

12. The equation

$$x_1\begin{bmatrix}1\\0\\1\end{bmatrix}+x_2\begin{bmatrix}-2\\3\\-2\end{bmatrix}+x_3\begin{bmatrix}-6\\7\\5\end{bmatrix}=\begin{bmatrix}11\\-5\\9\end{bmatrix} \quad (*)$$

$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$
$$\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$M=\begin{bmatrix}1&-2&-6&11\\0&3&7&-5\\1&-2&5&9\end{bmatrix}$$

Row reduce M until the pivot positions are visible:

$$M\sim\begin{bmatrix}\textcircled{1}&-2&-6&11\\0&\textcircled{3}&7&-5\\0&0&\textcircled{11}&-2\end{bmatrix}$$

The linear system corresponding to M *has* a solution, so the vector equation (*) has a solution, and therefore **b** *is* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

13. Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. To determine if **b** is a linear combination of these columns, use the boxed fact in the subsection *Linear Combinations*. Row reduce the augmented matrix $[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3\ \ \mathbf{b}]$ until you reach echelon form:

$$[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3\ \ \mathbf{b}]=\begin{bmatrix}1&-4&2&3\\0&3&5&-7\\-2&8&-4&-3\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&-4&2&3\\0&\textcircled{3}&5&-7\\0&0&0&\textcircled{3}\end{bmatrix}$$

The system for this augmented matrix is inconsistent, so **b** is *not* a linear combination of the columns of A.

14. Row reduce the augmented matrix $[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3\ \ \mathbf{b}]$ until you reach echelon form:

$$[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3\ \ \mathbf{b}]=\begin{bmatrix}1&0&5&2\\-2&1&-6&-1\\0&2&8&6\end{bmatrix}\sim\begin{bmatrix}1&0&5&2\\0&1&4&3\\0&2&8&6\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&0&5&2\\0&\textcircled{1}&4&3\\0&0&0&0\end{bmatrix}.$$

The linear system corresponding to this matrix *has* a solution, so **b** is a linear combination of the columns of A.

15. $$[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{b}]=\begin{bmatrix}1&-5&3\\3&-8&-5\\-1&2&h\end{bmatrix}\sim\begin{bmatrix}1&-5&3\\0&7&-14\\0&-3&h+3\end{bmatrix}\sim\begin{bmatrix}1&-5&3\\0&1&-2\\0&-3&h+3\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&-5&3\\0&\textcircled{1}&-2\\0&0&h-3\end{bmatrix}.$$ The vector **b** is in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$ when $h - 3$ is zero, that is, when $h = 3$.

16. $$[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{y}]=\begin{bmatrix}1&-2&h\\0&1&-3\\-2&7&-5\end{bmatrix}\sim\begin{bmatrix}1&-2&h\\0&1&-3\\0&3&-5+2h\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&-2&h\\0&\textcircled{1}&-3\\0&0&4+2h\end{bmatrix}.$$ The vector **y** is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ when $4 + 2h$ is zero, that is, when $h = -2$.

17. Noninteger weights are acceptable, of course, but some simple choices are $0\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \mathbf{0}$, and

$$1\cdot\mathbf{v}_1+0\cdot\mathbf{v}_2=\begin{bmatrix}3\\1\\2\end{bmatrix},\quad 0\cdot\mathbf{v}_1+1\cdot\mathbf{v}_2=\begin{bmatrix}-4\\0\\1\end{bmatrix},\quad 1\cdot\mathbf{v}_1+1\cdot\mathbf{v}_2=\begin{bmatrix}-1\\1\\3\end{bmatrix},\quad 1\cdot\mathbf{v}_1-1\cdot\mathbf{v}_2=\begin{bmatrix}7\\1\\1\end{bmatrix}$$

18. Some likely choices are $0\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \mathbf{0}$, and

$$1\cdot\mathbf{v}_1+0\cdot\mathbf{v}_2=\begin{bmatrix}1\\1\\-2\end{bmatrix},\quad 0\cdot\mathbf{v}_1+1\cdot\mathbf{v}_2=\begin{bmatrix}-2\\3\\0\end{bmatrix},\quad 1\cdot\mathbf{v}_1+1\cdot\mathbf{v}_2=\begin{bmatrix}-1\\4\\-2\end{bmatrix},\quad 1\cdot\mathbf{v}_1-1\cdot\mathbf{v}_2=\begin{bmatrix}3\\-2\\-2\end{bmatrix}$$

19. By inspection, $\mathbf{v}_2 = (3/2)\mathbf{v}_1$. Any linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ is actually just a multiple of $\mathbf{v}_1$. For instance,

$$a\mathbf{v}_1 + b\mathbf{v}_2 = a\mathbf{v}_1 + b(3/2)\mathbf{v}_1 = (a + 3b/2)\mathbf{v}_1$$

So Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is the set of points on the line through $\mathbf{v}_1$ and $\mathbf{0}$.

Note: Exercises 19 and 20 prepare the way for ideas in Sections 1.4 and 1.7.

20. Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane in $\mathbb{R}^3$ through the origin, because neither vector in this problem is a multiple of the other.

21. Let $\mathbf{y}=\begin{bmatrix}h\\k\end{bmatrix}$. Then $[\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{y}]=\begin{bmatrix}2&2&h\\-1&1&k\end{bmatrix}\sim\begin{bmatrix}\textcircled{2}&2&h\\0&\textcircled{2}&k+h/2\end{bmatrix}$. This augmented matrix corresponds to a consistent system for all h and k. So **y** is in Span$\{\mathbf{u}, \mathbf{v}\}$ for all h and k.

22. Construct any 3×4 matrix in echelon form that corresponds to an inconsistent system. Perform sufficient row operations on the matrix to eliminate all zero entries in the first three columns.

23. a. False. The alternative notation for a (column) vector is $(-4, 3)$, using parentheses and commas.

    b. False. Plot the points to verify this. Or, see the statement preceding Example 3. If $\begin{bmatrix}-5\\2\end{bmatrix}$ were on the line through $\begin{bmatrix}-2\\5\end{bmatrix}$ and the origin, then $\begin{bmatrix}-5\\2\end{bmatrix}$ would have to be a multiple of $\begin{bmatrix}-2\\5\end{bmatrix}$, which is not the case.

    c. True. See the line displayed just before Example 4.

    d. True. See the box that discusses the matrix in (5).

e. False. The statement is often true, but Span{**u**, **v**} is not a plane when **v** is a multiple of **u**, or when **u** is the zero vector.

24. **a.** False. Span{**u**, **v**} can be a plane.

b. True. See the beginning of the subsection *Vectors in* $\mathbf{R}^n$.

c. True. See the comment following the definition of Span{$\mathbf{v}_1, \ldots, \mathbf{v}_p$}.

d. False. $(\mathbf{u} - \mathbf{v}) + \mathbf{v} = \mathbf{u} - \mathbf{v} + \mathbf{v} = \mathbf{u}$.

e. False. Setting all the weights equal to zero results in a legitimate linear combination of a set of vectors.

25. **a.** There are only three vectors in the set {$\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$}, and **b** is not one of them.

b. There are infinitely many vectors in W = Span{$\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$}. To determine if **b** is in W, use the method of Exercise 13.

$$[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{b}] = \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -4 & 4 \\ 0 & \textcircled{3} & -2 & 1 \\ 0 & 0 & \textcircled{-1} & 2 \end{bmatrix}$$

The system for this augmented matrix is consistent, so **b** is in W.

c. $\mathbf{a}_1 = 1\mathbf{a}_1 + 0\mathbf{a}_2 + 0\mathbf{a}_3$. See the discussion in the text following the definition of Span{$\mathbf{v}_1, \ldots, \mathbf{v}_p$}.

26. **a.** $$[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{b}] = \begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 8 & 8 & 8 \\ 0 & -2 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 8 & 8 & 8 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

No, **b** is not a linear combination of the columns of A, that is, **b** is not in W.

b. The second column of A is in W because $\mathbf{a}_2 = 0 \cdot \mathbf{a}_1 + 1 \cdot \mathbf{a}_2 + 0 \cdot \mathbf{a}_3$.

27. **a.** $5\mathbf{v}_1$ is the output of 5 days' operation of mine #1.

b. The total output is $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$, so x_1 and x_2 should satisfy $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \begin{bmatrix} 240 \\ 2824 \end{bmatrix}$.

c. **[M]** Reduce the augmented matrix $\begin{bmatrix} 30 & 40 & 240 \\ 600 & 380 & 2824 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.73 \\ 0 & 1 & 4.70 \end{bmatrix}$.

Operate mine #1 for 1.73 days and mine #2 for 4.70 days. (This is an approximate solution.)

28. **a.** The amount of heat produced when the steam plant burns x_1 tons of anthracite and x_2 tons of bituminous coal is $27.6x_1 + 30.2x_2$ million Btu.

b. The total output produced by x_1 tons of anthracite and x_2 tons of bituminous coal is given by the vector $x_1\begin{bmatrix} 27.6 \\ 3100 \\ 250 \end{bmatrix} + x_2\begin{bmatrix} 30.2 \\ 6400 \\ 360 \end{bmatrix}$.

c. **[M]** The appropriate values for x_1 and x_2 satisfy $x_1\begin{bmatrix} 27.6 \\ 3100 \\ 250 \end{bmatrix} + x_2\begin{bmatrix} 30.2 \\ 6400 \\ 360 \end{bmatrix} = \begin{bmatrix} 162 \\ 23{,}610 \\ 1{,}623 \end{bmatrix}$.

To solve, row reduce the augmented matrix:

$$\begin{bmatrix} 27.6 & 30.2 & 162 \\ 3100 & 6400 & 23610 \\ 250 & 360 & 1623 \end{bmatrix} \sim \begin{bmatrix} 1.000 & 0 & 3.900 \\ 0 & 1.000 & 1.800 \\ 0 & 0 & 0 \end{bmatrix}$$

The steam plant burned 3.9 tons of anthracite coal and 1.8 tons of bituminous coal.

29. The total mass is 4 + 2 + 3 + 5 = 14. So $\mathbf{v} = (4\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3 + 5\mathbf{v}_4)/14$. That is,

$$\mathbf{v} = \frac{1}{14}\left(4\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} + 2\begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix} + 5\begin{bmatrix} 1 \\ -6 \\ 0 \end{bmatrix}\right) = \frac{1}{14}\begin{bmatrix} 8-8+12+5 \\ -8+4+0-30 \\ 16+6-6+0 \end{bmatrix} = \begin{bmatrix} 17/14 \\ -17/7 \\ 8/7 \end{bmatrix} \approx \begin{bmatrix} 1.214 \\ -2.429 \\ 1.143 \end{bmatrix}$$

30. Let m be the total mass of the system. By definition,

$$\mathbf{v} = \frac{1}{m}(m_1\mathbf{v}_1 + \cdots + m_k\mathbf{v}_k) = \frac{m_1}{m}\mathbf{v}_1 + \cdots + \frac{m_k}{m}\mathbf{v}_k$$

The second expression displays **v** as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, which shows that **v** is in Span{$\mathbf{v}_1, \ldots, \mathbf{v}_k$}.

31. **a.** The center of mass is $\frac{1}{3}\left(1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 8 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 10/3 \\ 2 \end{bmatrix}$.

b. The total mass of the new system is 9 grams. The three masses added, w_1, w_2, and w_3, satisfy the equation

$$\frac{1}{9}\left((w_1+1) \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (w_2+1) \cdot \begin{bmatrix} 8 \\ 1 \end{bmatrix} + (w_3+1) \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

which can be rearranged to

$$(w_1+1) \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (w_2+1) \cdot \begin{bmatrix} 8 \\ 1 \end{bmatrix} + (w_3+1) \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 18 \\ 18 \end{bmatrix}$$

and

$$w_1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + w_2 \cdot \begin{bmatrix} 8 \\ 1 \end{bmatrix} + w_3 \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$$

The condition $w_1 + w_2 + w_3 = 6$ and the vector equation above combine to produce a system of three equations whose augmented matrix is shown below, along with a sequence of row operations:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 8 & 2 & 8 \\ 1 & 1 & 4 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 8 & 2 & 8 \\ 0 & 0 & 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 8 & 2 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 8 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3.5 \\ 0 & 8 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3.5 \\ 0 & 1 & 0 & .5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Answer: Add 3.5 g at (0, 1), add .5 g at (8, 1), and add 2 g at (2, 4).

Extra problem: Ignore the mass of the plate, and distribute 6 gm at the three vertices to make the center of mass at (2, 2). Answer: Place 3 g at (0, 1), 1 g at (8, 1), and 2 g at (2, 4).

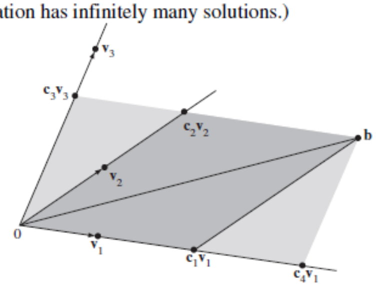
32. See the parallelograms drawn on the figure from the text that accompanies this exercise. Here c_1, c_2, c_3, and c_4 are suitable scalars. The darker parallelogram shows that **b** is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, that is

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0\cdot\mathbf{v}_3 = \mathbf{b}$$

The larger parallelogram shows that **b** is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$, that is,

$$c_4\mathbf{v}_1 + 0\cdot\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b}$$

So the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$ has at least two solutions, not just one solution. (In fact, the equation has infinitely many solutions.)



33. **a.** For $j = 1,\dots, n$, the jth entry of $(\mathbf{u} + \mathbf{v}) + \mathbf{w}$ is $(u_j + v_j) + w_j$. By associativity of addition in **R**, this entry equals $u_j + (v_j + w_j)$, which is the jth entry of $\mathbf{u} + (\mathbf{v} + \mathbf{w})$. By definition of equality of vectors, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

b. For any scalar c, the jth entry of $c(\mathbf{u} + \mathbf{v})$ is $c(u_j + v_j)$, and the jth entry of $c\mathbf{u} + c\mathbf{v}$ is $cu_j + cv_j$ (by definition of scalar multiplication and vector addition). These entries are equal, by a distributive law in **R**. So $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

34. **a.** For $j = 1,\dots, n$, $u_j + (-1)u_j = (-1)u_j + u_j = 0$, by properties of **R**. By vector equality,

$$\mathbf{u} + (-1)\mathbf{u} = (-1)\mathbf{u} + \mathbf{u} = \mathbf{0}.$$

b. For scalars c and d, the jth entries of $c(d\mathbf{u})$ and $(cd)\mathbf{u}$ are $c(du_j)$ and $(cd)u_j$, respectively. These entries in **R** are equal, so the vectors $c(d\mathbf{u})$ and $(cd)\mathbf{u}$ are equal.