Math 260 Homework 1.2

1. Reduced echelon form: a and b. Echelon form: d. Not echelon: c.

2. Reduced echelon form: a. Echelon form: b and d. Not echelon: c.

3. $$\begin{bmatrix}1&2&4&8\\2&4&6&8\\3&6&9&12\end{bmatrix}\sim\begin{bmatrix}1&2&4&8\\0&0&-2&-8\\0&0&-3&-12\end{bmatrix}\sim\begin{bmatrix}1&2&4&8\\0&0&1&4\\0&0&-3&-12\end{bmatrix}$$

$$\sim\begin{bmatrix}1&2&4&8\\0&0&1&4\\0&0&0&0\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&2&0&-8\\0&0&\textcircled{1}&4\\0&0&0&0\end{bmatrix}.\ \text{Pivot cols 1 and 3.}\ \begin{bmatrix}\textcircled{1}&2&4&8\\2&4&\textcircled{6}&8\\3&6&9&12\end{bmatrix}$$

4. $$\begin{bmatrix}1&2&4&5\\2&4&5&4\\4&5&4&2\end{bmatrix}\sim\begin{bmatrix}1&2&4&5\\0&0&-3&-6\\0&-3&-12&-18\end{bmatrix}\sim\begin{bmatrix}1&2&4&5\\0&-3&-12&-18\\0&0&-3&-6\end{bmatrix}\sim\begin{bmatrix}1&2&4&5\\0&1&4&6\\0&0&-3&-6\end{bmatrix}$$

$$\sim\begin{bmatrix}1&2&4&5\\0&1&4&6\\0&0&1&2\end{bmatrix}\sim\begin{bmatrix}1&2&4&5\\0&1&0&-2\\0&0&1&2\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&0&0&1\\0&\textcircled{1}&0&-2\\0&0&\textcircled{1}&2\end{bmatrix}.\ \text{Pivot cols 1, 2, and 3}\ \begin{bmatrix}\textcircled{1}&2&4&5\\2&\textcircled{4}&5&4\\4&5&\textcircled{4}&2\end{bmatrix}$$

5. $$\begin{bmatrix}\blacksquare&*\\0&\blacksquare\end{bmatrix},\begin{bmatrix}\blacksquare&*\\0&0\end{bmatrix},\begin{bmatrix}0&\blacksquare\\0&0\end{bmatrix}$$

6. $$\begin{bmatrix}\blacksquare&*\\0&\blacksquare\\0&0\end{bmatrix},\begin{bmatrix}\blacksquare&*\\0&0\\0&0\end{bmatrix},\begin{bmatrix}0&\blacksquare\\0&0\\0&0\end{bmatrix}$$

7. $$\begin{bmatrix}1&3&4&7\\3&9&7&6\end{bmatrix}\sim\begin{bmatrix}1&3&4&7\\0&0&-5&-15\end{bmatrix}\sim\begin{bmatrix}1&3&4&7\\0&0&1&3\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&3&0&-5\\0&0&\textcircled{1}&3\end{bmatrix}$$

Corresponding system of equations:
$$\begin{aligned}\textcircled{x_1} + 3x_2 &= -5\\ \textcircled{x_3} &= 3\end{aligned}$$

The basic variables (corresponding to the pivot positions) are x_1 and x_3. The remaining variable x_2 is free. Solve for the basic variables in terms of the free variable. The general solution is

$$\begin{cases}x_1 = -5 - 3x_2\\ x_2 \text{ is free}\\ x_3 = 3\end{cases}$$

8. $$\begin{bmatrix}1&-3&0&-5\\-3&7&0&9\end{bmatrix}\sim\begin{bmatrix}1&-3&0&-5\\0&-2&0&-6\end{bmatrix}\sim\begin{bmatrix}1&-3&0&-5\\0&1&0&3\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&0&0&4\\0&\textcircled{1}&0&3\end{bmatrix}$$

Corresponding system of equations:
$$\begin{aligned}\textcircled{x_1} &= 4\\ \textcircled{x_2} &= 3\end{aligned}$$

The basic variables (corresponding to the pivot positions) are x_1 and x_2. The remaining variable x_3 is free. Solve for the basic variables in terms of the free variable. In this particular problem, the basic variables do not depend on the value of the free variable.

General solution: $$\begin{cases}x_1 = 4\\ x_2 = 3\\ x_3 \text{ is free}\end{cases}$$

Note: A common error in Exercise 8 is to assume that x_3 is zero. To avoid this, identify the basic variables first. Any remaining variables are *free*. (This type of computation will arise in Chapter 5.)

9. $$\begin{bmatrix}0&1&-2&3\\1&-3&4&-6\end{bmatrix}\sim\begin{bmatrix}1&-3&4&-6\\0&1&-2&3\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&0&-2&3\\0&\textcircled{1}&-2&3\end{bmatrix}$$

Corresponding system:
$$\begin{aligned}\textcircled{x_1} - 2x_3 &= 3\\ \textcircled{x_2} - 2x_3 &= 3\end{aligned}$$

Basic variables: x_1, x_2; free variable: x_3. General solution: $$\begin{cases}x_1 = 3 + 2x_3\\ x_2 = 3 + 2x_3\\ x_3 \text{ is free}\end{cases}$$

10. $$\begin{bmatrix}1&-2&-1&4\\-2&4&-5&6\end{bmatrix}\sim\begin{bmatrix}1&-2&-1&4\\0&0&-7&14\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&-2&0&2\\0&0&\textcircled{1}&-2\end{bmatrix}$$

Corresponding system:
$$\begin{aligned}\textcircled{x_1} - 2x_2 &= 2\\ \textcircled{x_3} &= -2\end{aligned}$$

Basic variables: x_1, x_3; free variable: x_2. General solution: $$\begin{cases}x_1 = 2 + 2x_2\\ x_2 \text{ is free}\\ x_3 = -2\end{cases}$$

11. $$\begin{bmatrix}3&-2&4&0\\9&-6&12&0\\6&-4&8&0\end{bmatrix}\sim\begin{bmatrix}3&-2&4&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&-2/3&4/3&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

Corresponding system:
$$\begin{aligned}\textcircled{x_1} - \frac{2}{3}x_2 + \frac{4}{3}x_3 &= 0\\ 0 &= 0\\ 0 &= 0\end{aligned}$$

Basic variable: x_1; free variables x_2, x_3. General solution: $\begin{cases} x_1 = \frac{2}{3}x_2 - \frac{4}{3}x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$

12. Since the bottom row of the matrix is equivalent to the equation 0 = 1, the system has no solutions.

13. $$\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 0 & 9 & 2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 0 & -3 & 5 \\ 0 & \textcircled{1} & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Corresponding system: $$\begin{aligned} \textcircled{x_1} \quad\quad - 3x_5 &= 5 \\ \textcircled{x_2} \quad - 4x_5 &= 1 \\ \textcircled{x_4} + 9x_5 &= 4 \\ 0 &= 0 \end{aligned}$$

Basic variables: x_1, x_2, x_4; free variables: x_3, x_5. General solution: $\begin{cases} x_1 = 5 + 3x_5 \\ x_2 = 1 + 4x_5 \\ x_3 \text{ is free} \\ x_4 = 4 - 9x_5 \\ x_5 \text{ is free} \end{cases}$

Note: The *Study Guide* discusses the common mistake $x_3 = 0$.

14. $$\begin{bmatrix} 1 & 0 & -5 & 0 & -8 & 3 \\ 0 & 1 & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -5 & 0 & 0 & 3 \\ 0 & \textcircled{1} & 4 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Corresponding system: $$\begin{aligned} \textcircled{x_1} \quad - 5x_3 \quad\quad &= 3 \\ \textcircled{x_2} + 4x_3 - x_4 &= 6 \\ \textcircled{x_5} &= 0 \\ 0 &= 0 \end{aligned}$$

Basic variables: x_1, x_2, x_5; free variables: x_3, x_4. General solution: $\begin{cases} x_1 = 3 + 5x_3 \\ x_2 = 6 - 4x_3 + x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \\ x_5 = 0 \end{cases}$

15. **a.** The system is consistent. There are many solutions because x_3 is a free variable.

b. The system is consistent. There are many solutions because x_1 is a free variable.

16. **a.** The system is inconsistent. (The rightmost column of the augmented matrix is a pivot column).

b. The system is consistent. There are many solutions because x_2 is a free variable.

17. $\begin{bmatrix} 1 & -1 & 4 \\ -2 & 3 & h \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -1 & 4 \\ 0 & \textcircled{1} & h+8 \end{bmatrix}$ The system has a solution for all values of h since the augmented column cannot be a pivot column.

18. $\begin{bmatrix} 1 & -3 & 1 \\ h & 6 & -2 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 1 \\ 0 & 3h+6 & -h-2 \end{bmatrix}$ If $3h + 6$ is zero, that is, if $h = -2$, then the system has a solution, because 0 equals 0. When $h \neq -2$, the system has a solution since the augmented column cannot be a pivot column. Thus the system has a solution for all values of h.

19. $\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix}$

a. When $h = 2$ and $k \neq 8$, the augmented column is a pivot column, and the system is inconsistent.

b. When $h \neq 2$, the system is consistent and has a unique solution. There are no free variables.

c. When $h = 2$ and $k = 8$, the system is consistent and has many solutions.

20. $\begin{bmatrix} 1 & -3 & 1 \\ 2 & h & k \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 1 \\ 0 & h+6 & k-2 \end{bmatrix}$

a. When $h = -6$ and $k \neq 2$, the system is inconsistent, because the augmented column is a pivot column.

b. When $h \neq -6$, the system is consistent and has a unique solution. There are no free variables.

c. When $h = -6$ and $k = 2$, the system is consistent and has many solutions.

21. **a.** False. See Theorem 1.

b. False. See the second paragraph of the section.

c. True. Basic variables are defined after equation (4).

d. True. This statement is at the beginning of *Parametric Descriptions of Solution Sets*.

e. False. The row shown corresponds to the equation $5x_4 = 0$, which does not by itself lead to a contradiction. So the system might be consistent or it might be inconsistent.

22. **a.** True. See Theorem 1.

b. False. See Theorem 2.

c. False. See the beginning of the subsection *Pivot Positions*. The pivot positions in a matrix are determined completely by the positions of the leading entries in the nonzero rows of any echelon form obtained from the matrix.

d. True. See the paragraph just before Example 4.

e. False. The existence of at least one solution is not related to the presence or absence of free variables. If the system is inconsistent, the solution set is empty. See the solution of Practice Problem 2.

23. Since there are four pivots (one in each row), the augmented matrix must reduce to the form

$$\begin{bmatrix} \textcircled{1} & 0 & 0 & 0 & a \\ 0 & \textcircled{1} & 0 & 0 & b \\ 0 & 0 & \textcircled{1} & 0 & c \\ 0 & 0 & 0 & \textcircled{1} & d \end{bmatrix} \text{ and so } \begin{array}{rcl} \textcircled{x_1} & = & a \\ \textcircled{x_2} & = & b \\ \textcircled{x_3} & = & c \\ \textcircled{x_4} & = & d \end{array}$$

No matter what the values of a, b, c, and d, the solution exists and is unique.

24. The system is consistent because there is not a pivot in column 5, which means that there is not a row of the form $[0 \ \ 0 \ \ 0 \ \ 0 \ \ 1]$. Since the matrix is the *augmented* matrix for a system, Theorem 2 shows that the system has a solution.

25. If the coefficient matrix has a pivot position in every row, then there is a pivot position in the bottom row, and there is no room for a pivot in the augmented column. So, the system is consistent, by Theorem 2.

26. Since the coefficient matrix has three pivot columns, there is a pivot in each row of the coefficient matrix. Thus the augmented matrix will not have a row of the form $[0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 1]$, and the system is consistent.

27. *"If a linear system is consistent, then the solution is unique if and only if every column in the coefficient matrix is a pivot column; otherwise there are infinitely many solutions. "*

This statement is true because the free variables correspond to *nonpivot* columns of the coefficient matrix. The columns are all pivot columns if and only if there are no free variables. And there are no free variables if and only if the solution is unique, by Theorem 2.

28. Every column in the augmented matrix *except the rightmost column* is a pivot column, and the rightmost column is *not* a pivot column.

29. An underdetermined system always has more variables than equations. There cannot be more basic variables than there are equations, so there must be at least one free variable. Such a variable may be assigned infinitely many different values. If the system is consistent, each different value of a free variable will produce a different solution, and the system will not have a unique solution. If the system is inconsistent, it will not have any solution.

30. Example: $$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 4 \\ 2x_1 & + & 2x_2 & + & 2x_3 & = & 5 \end{array}$$

31. Yes, a system of linear equations with more equations than unknowns can be consistent.

Example (in which $x_1 = x_2 = 1$): $$\begin{array}{rcrcl} x_1 & + & x_2 & = & 2 \\ x_1 & - & x_2 & = & 0 \\ 3x_1 & + & 2x_2 & = & 5 \end{array}$$

32. According to the numerical note in Section 1.2, when $n = 20$ the reduction to echelon form takes about $2(20)^3/3 \approx 5{,}333$ flops, while further reduction to reduced echelon form needs at most $(20)^2 = 400$ flops. Of the total flops, the "backward phase" is about $400/5733 = .07$ or about 7%. When $n = 200$, the estimates are $2(200)^3/3 \approx 5{,}333{,}333$ flops for the reduction to echelon form and $(200)^2 = 40{,}000$ flops for the backward phase. The fraction associated with the backward phase is about $(4\times10^4)/(5.3\times10^6) = .007$, or about .7%.

33. For a quadratic polynomial $p(t) = a_0 + a_1t + a_2t^2$ to exactly fit the data (1, 6), (2, 15), and (3, 28), the coefficients a_0, a_1, a_2 must satisfy the systems of equations given in the text. Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 4 & 15 \\ 1 & 3 & 9 & 28 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 9 \\ 0 & 2 & 8 & 22 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 1 \\ 0 & \textcircled{1} & 0 & 3 \\ 0 & 0 & \textcircled{1} & 2 \end{bmatrix}$$

The polynomial is $p(t) = 1 + 3t + 2t^2$.

34. **[M]** The system of equations to be solved is:

$$\begin{array}{lclclclclclcl} a_0 & + & a_1\cdot 0 & + & a_2\cdot 0^2 & + & a_3\cdot 0^3 & + & a_4\cdot 0^4 & + & a_5\cdot 0^5 & = & 0 \\ a_0 & + & a_1\cdot 2 & + & a_2\cdot 2^2 & + & a_3\cdot 2^3 & + & a_4\cdot 2^4 & + & a_5\cdot 2^5 & = & 2.90 \\ a_0 & + & a_1\cdot 4 & + & a_2\cdot 4^2 & + & a_3\cdot 4^3 & + & a_4\cdot 4^4 & + & a_5\cdot 4^5 & = & 14.8 \\ a_0 & + & a_1\cdot 6 & + & a_2\cdot 6^2 & + & a_3\cdot 6^3 & + & a_4\cdot 6^4 & + & a_5\cdot 6^5 & = & 39.6 \\ a_0 & + & a_1\cdot 8 & + & a_2\cdot 8^2 & + & a_3\cdot 8^3 & + & a_4\cdot 8^4 & + & a_5\cdot 8^5 & = & 74.3 \\ a_0 & + & a_1\cdot 10 & + & a_2\cdot 10^2 & + & a_3\cdot 10^3 & + & a_4\cdot 10^4 & + & a_5\cdot 10^5 & = & 119 \end{array}$$

The unknowns are $a_0, a_1, \ldots, a_5$. Use technology to compute the reduced echelon of the augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 16 & 32 & 2.9 \\ 1 & 4 & 16 & 64 & 256 & 1024 & 14.8 \\ 1 & 6 & 36 & 216 & 1296 & 7776 & 39.6 \\ 1 & 8 & 64 & 512 & 4096 & 32768 & 74.3 \\ 1 & 10 & 10^2 & 10^3 & 10^4 & 10^5 & 119 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 16 & 32 & 2.9 \\ 0 & 0 & 8 & 48 & 224 & 960 & 9 \\ 0 & 0 & 24 & 192 & 1248 & 7680 & 30.9 \\ 0 & 0 & 48 & 480 & 4032 & 32640 & 62.7 \\ 0 & 0 & 80 & 960 & 9920 & 99840 & 104.5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 16 & 32 & 2.9 \\ 0 & 0 & 8 & 48 & 224 & 960 & 9 \\ 0 & 0 & 0 & 48 & 576 & 4800 & 3.9 \\ 0 & 0 & 0 & 192 & 2688 & 26880 & 8.7 \\ 0 & 0 & 0 & 480 & 7680 & 90240 & 14.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 16 & 32 & 2.9 \\ 0 & 0 & 8 & 48 & 224 & 960 & 9 \\ 0 & 0 & 0 & 48 & 576 & 4800 & 3.9 \\ 0 & 0 & 0 & 0 & 384 & 7680 & -6.9 \\ 0 & 0 & 0 & 0 & 1920 & 42240 & -24.5 \end{bmatrix}$$