Math 260 Homework 1.1

1. $\begin{aligned} x_1 + 5x_2 &= 7 \\ -2x_1 - 7x_2 &= -5 \end{aligned}$ $\begin{bmatrix} 1 & 5 & 7 \\ -2 & -7 & -5 \end{bmatrix}$

Replace R2 by R2 + (2)R1 and obtain: $\begin{aligned} x_1 + 5x_2 &= 7 \\ 3x_2 &= 9 \end{aligned}$ $\begin{bmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{bmatrix}$

Scale R2 by 1/3: $\begin{aligned} x_1 + 5x_2 &= 7 \\ x_2 &= 3 \end{aligned}$ $\begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 3 \end{bmatrix}$

Replace R1 by R1 + (–5)R2: $\begin{aligned} x_1 \quad &= -8 \\ x_2 &= 3 \end{aligned}$ $\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \end{bmatrix}$

The solution is $(x_1, x_2) = (-8, 3)$, or simply (–8, 3).

2. $\begin{aligned} 3x_1 + 6x_2 &= -3 \\ 5x_1 + 7x_2 &= 10 \end{aligned}$ $\begin{bmatrix} 3 & 6 & -3 \\ 5 & 7 & 10 \end{bmatrix}$

Scale R1 by 1/3 and obtain: $\begin{aligned} x_1 + 2x_2 &= -1 \\ 5x_1 + 7x_2 &= 10 \end{aligned}$ $\begin{bmatrix} 1 & 2 & -1 \\ 5 & 7 & 10 \end{bmatrix}$

Replace R2 by R2 + (–5)R1: $\begin{aligned} x_1 + 2x_2 &= -1 \\ -3x_2 &= 15 \end{aligned}$ $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 15 \end{bmatrix}$

Scale R2 by –1/3: $\begin{aligned} x_1 + 2x_2 &= -1 \\ x_2 &= -5 \end{aligned}$ $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -5 \end{bmatrix}$

Replace R1 by R1 + (–2)R2: $\begin{aligned} x_1 \quad &= 9 \\ x_2 &= -5 \end{aligned}$ $\begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -5 \end{bmatrix}$

The solution is $(x_1, x_2) = (9, -5)$, or simply (9, –5).

3. The point of intersection satisfies the system of two linear equations:

$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_1 - x_2 &= 1 \end{aligned}$ $\begin{bmatrix} 1 & 2 & 4 \\ 1 & -1 & 1 \end{bmatrix}$

Replace R2 by R2 + (–1)R1 and obtain: $\begin{aligned} x_1 + 2x_2 &= 4 \\ -3x_2 &= -3 \end{aligned}$ $\begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -3 \end{bmatrix}$

Scale R2 by –1/3: $\begin{aligned} x_1 + 2x_2 &= 4 \\ x_2 &= 1 \end{aligned}$ $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix}$

Replace R1 by R1 + (–2)R2: $\begin{aligned} x_1 \quad &= 2 \\ x_2 &= 1 \end{aligned}$ $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

The point of intersection is $(x_1, x_2) = (2, 1)$.

4. The point of intersection satisfies the system of two linear equations:

$\begin{aligned} x_1 + 2x_2 &= -13 \\ 3x_1 - 2x_2 &= 1 \end{aligned}$ $\begin{bmatrix} 1 & 2 & -13 \\ 3 & -2 & 1 \end{bmatrix}$

Replace R2 by R2 + (–3)R1 and obtain: $\begin{aligned} x_1 + 2x_2 &= -13 \\ -8x_2 &= 40 \end{aligned}$ $\begin{bmatrix} 1 & 2 & -13 \\ 0 & -8 & 40 \end{bmatrix}$

Scale R2 by –1/8: $\begin{aligned} x_1 + 2x_2 &= -13 \\ x_2 &= -5 \end{aligned}$ $\begin{bmatrix} 1 & 2 & -13 \\ 0 & 1 & -5 \end{bmatrix}$

Replace R1 by R1 + (–2)R2: $\begin{aligned} x_1 \quad &= -3 \\ x_2 &= -5 \end{aligned}$ $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -5 \end{bmatrix}$

The point of intersection is $(x_1, x_2) = (-3, -5)$.

5. The system is already in "triangular" form. The fourth equation is $x_4 = -5$, and the other equations do not contain the variable x_4. The next two steps should be to use the variable x_3 in the third equation to eliminate that variable from the first two equations. In matrix notation, that means to replace R2 by its sum with –4 times R3, and then replace R1 by its sum with 3 times R3.

6. One more step will put the system in triangular form. Replace R4 by its sum with –4 times R3, which produces $\begin{bmatrix} 1 & -6 & 4 & 0 & -1 \\ 0 & 2 & -7 & 0 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & -7 & 14 \end{bmatrix}$. After that, the next step is to scale the fourth row by –1/7.

7. Ordinarily, the next step would be to interchange R3 and R4, to put a 1 in the third row and third column. But in this case, the third row of the augmented matrix corresponds to the equation $0\,x_1 + 0\,x_2 + 0\,x_3 = 1$, or simply, 0 = 1. A system containing this condition has no solution. Further row operations are unnecessary once an equation such as 0 = 1 is evident. The solution set is empty.

8. The standard row operations are:

$$\begin{bmatrix}1&-5&4&0&0\\0&1&0&1&0\\0&0&3&0&0\\0&0&0&2&0\end{bmatrix}\sim\begin{bmatrix}1&-5&4&0&0\\0&1&0&1&0\\0&0&3&0&0\\0&0&0&1&0\end{bmatrix}\sim\begin{bmatrix}1&-5&4&0&0\\0&1&0&0&0\\0&0&3&0&0\\0&0&0&1&0\end{bmatrix}\sim\begin{bmatrix}1&-5&4&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{bmatrix}$$

$$\sim\begin{bmatrix}1&-5&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{bmatrix}$$

The solution set contains one solution: (0, 0, 0, 0).

9. The system has already been reduced to triangular form. Begin by replacing R3 by R3 + (3)R4:

$$\begin{bmatrix}1&-1&0&0&-5\\0&1&-2&0&-7\\0&0&1&-3&2\\0&0&0&1&4\end{bmatrix}\sim\begin{bmatrix}1&-1&0&0&-5\\0&1&-2&0&-7\\0&0&1&0&14\\0&0&0&1&4\end{bmatrix}$$

Next, replace R2 by R2 + (2)R3. Finally, replace R1 by R1 + R2:

$$\sim\begin{bmatrix}1&-1&0&0&-5\\0&1&0&0&21\\0&0&1&0&14\\0&0&0&1&4\end{bmatrix}\sim\begin{bmatrix}1&0&0&0&16\\0&1&0&0&21\\0&0&1&0&14\\0&0&0&1&4\end{bmatrix}$$

The solution set contains one solution: (16, 21, 14, 4).

10. The system has already been reduced to triangular form. Use the 1 in the fourth row to change the 3 and –2 above it to zeros. That is, replace R2 by R2 + (-3)R4 and replace R1 by R1 + (2)R4. For the final step, replace R1 by R1 + (-3)R2.

$$\begin{bmatrix}1&3&0&-2&-7\\0&1&0&3&6\\0&0&1&0&2\\0&0&0&1&-2\end{bmatrix}\sim\begin{bmatrix}1&3&0&0&-11\\0&1&0&0&12\\0&0&1&0&2\\0&0&0&1&-2\end{bmatrix}\sim\begin{bmatrix}1&0&0&0&-47\\0&1&0&0&12\\0&0&1&0&2\\0&0&0&1&-2\end{bmatrix}$$

The solution set contains one solution: (–47, 12, 2, –2).

11. First, swap R1 and R2. Then replace R3 by R3 + (–2)R1. Finally, replace R3 by R3 + (1)R2.

$$\begin{bmatrix}0&1&5&-4\\1&4&3&-2\\2&7&1&-2\end{bmatrix}\sim\begin{bmatrix}1&4&3&-2\\0&1&5&-4\\2&7&1&-2\end{bmatrix}\sim\begin{bmatrix}1&4&3&-2\\0&1&5&-4\\0&-1&-5&2\end{bmatrix}\sim\begin{bmatrix}1&4&3&-2\\0&1&5&-4\\0&0&0&-2\end{bmatrix}$$

The system is inconsistent, because the last row would require that 0 = –2 if there were a solution. The solution set is empty.

12. Replace R2 by R2 + (–2)R1 and replace R3 by R3 + (2)R1. Finally, replace R3 by R3 + (3)R2.

$$\begin{bmatrix}1&-5&4&-3\\2&-7&3&-2\\-2&1&7&-1\end{bmatrix}\sim\begin{bmatrix}1&-5&4&-3\\0&3&-5&4\\0&-9&15&-7\end{bmatrix}\sim\begin{bmatrix}1&-5&4&-3\\0&3&-5&4\\0&0&0&5\end{bmatrix}$$

The system is inconsistent, because the last row would require that 0 = 5 if there were a solution. The solution set is empty.

13. $$\begin{bmatrix}1&0&-3&8\\2&2&9&7\\0&1&5&-2\end{bmatrix}\sim\begin{bmatrix}1&0&-3&8\\0&2&15&-9\\0&1&5&-2\end{bmatrix}\sim\begin{bmatrix}1&0&-3&8\\0&1&5&-2\\0&2&15&-9\end{bmatrix}\sim\begin{bmatrix}1&0&-3&8\\0&1&5&-2\\0&0&5&-5\end{bmatrix}$$

$$\sim\begin{bmatrix}1&0&-3&8\\0&1&5&-2\\0&0&1&-1\end{bmatrix}\sim\begin{bmatrix}1&0&0&5\\0&1&0&3\\0&0&1&-1\end{bmatrix}$$. The solution is (5, 3, –1).

14. $$\begin{bmatrix}2&0&-6&-8\\0&1&2&3\\3&6&-2&-4\end{bmatrix}\sim\begin{bmatrix}1&0&-3&-4\\0&1&2&3\\3&6&-2&-4\end{bmatrix}\sim\begin{bmatrix}1&0&-3&-4\\0&1&2&3\\0&6&7&8\end{bmatrix}\sim\begin{bmatrix}1&0&-3&-4\\0&1&2&3\\0&0&-5&-10\end{bmatrix}$$

$$\sim\begin{bmatrix}1&0&-3&-4\\0&1&2&3\\0&0&1&2\end{bmatrix}\sim\begin{bmatrix}1&0&-3&-4\\0&1&0&-1\\0&0&1&2\end{bmatrix}\sim\begin{bmatrix}1&0&0&2\\0&1&0&-1\\0&0&1&2\end{bmatrix}$$. The solution is (2, –1, 2).

15. First, replace R3 by R3 + (1)R1, then replace R4 by R4 + (1)R2, and finally replace R4 by R4 + (–1)R3.

$$\begin{bmatrix}1&-6&0&0&5\\0&1&-4&1&0\\-1&6&1&5&3\\0&-1&5&4&0\end{bmatrix}\sim\begin{bmatrix}1&-6&0&0&5\\0&1&-4&1&0\\0&0&1&5&8\\0&-1&5&4&0\end{bmatrix}\sim\begin{bmatrix}1&-6&0&0&5\\0&1&-4&1&0\\0&0&1&5&8\\0&0&1&5&0\end{bmatrix}\sim\begin{bmatrix}1&-6&0&0&5\\0&1&-4&1&0\\0&0&1&5&8\\0&0&0&0&-8\end{bmatrix}$$

The system is inconsistent, because the last row would require that 0 = –8 if there were a solution.

16. First replace R4 by R4 + (3/2)R1 and replace R4 by R4 + (–2/3)R2. (One could also scale R1 and R2 before adding to R4, but the arithmetic is rather easy keeping R1 and R2 unchanged.) Finally, replace R4 by R4 + (–1)R3.

$$\begin{bmatrix}2&0&0&-4&-10\\0&3&3&0&0\\0&0&1&4&-1\\-3&2&3&1&5\end{bmatrix}\sim\begin{bmatrix}2&0&0&-4&-10\\0&3&3&0&0\\0&0&1&4&-1\\0&2&3&-5&-10\end{bmatrix}\sim\begin{bmatrix}2&0&0&-4&-10\\0&3&3&0&0\\0&0&1&4&-1\\0&0&1&-5&-10\end{bmatrix}\sim\begin{bmatrix}2&0&0&-4&-10\\0&3&3&0&0\\0&0&1&4&-1\\0&0&0&-9&-9\end{bmatrix}$$

The system is now in triangular form and has a solution. In fact, using the argument from Example 2, one can see that the solution is unique.

17. Row reduce the augmented matrix corresponding to the given system of three equations:

$$\begin{bmatrix} 2 & 3 & -1 \\ 6 & 5 & 0 \\ 2 & -5 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & -4 & 3 \\ 0 & -8 & 8 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & -4 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

The third equation, $0 = 2$, shows that the system is inconsistent, so the three lines have no point in common.

18. Row reduce the augmented matrix corresponding to the given system of three equations:

$$\begin{bmatrix} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 2 & 3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -4 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & 4 & 4 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -6 & -6 \end{bmatrix}$$

The system is consistent, and using the argument from Example 2, there is only one solution. So the three planes have only one point in common.

19. $\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & h & 4 \\ 0 & 6-3h & -4 \end{bmatrix}$ Write c for $6 - 3h$. If $c = 0$, that is, if $h = 2$, then the system has no solution, because 0 cannot equal –4. Otherwise, when $h \neq 2$, the system has a solution.

20. $\begin{bmatrix} 1 & h & -5 \\ 2 & -8 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & h & -5 \\ 0 & -8-2h & 16 \end{bmatrix}$ Write c for $-8-2h$. If $c = 0$, that is, if $h = -4$, then the system has no solution, because 0 cannot equal 16. Otherwise, when $h \neq -4$, the system has a solution.

21. $\begin{bmatrix} 1 & 4 & -2 \\ 3 & h & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -2 \\ 0 & h-12 & 0 \end{bmatrix}$ Write c for $h-12$. Then the second equation $cx_2 = 0$ has a solution for every value of c. So the system is consistent for all h.

22. $\begin{bmatrix} -4 & 12 & h \\ 2 & -6 & -3 \end{bmatrix} \sim \begin{bmatrix} -4 & 12 & h \\ 0 & 0 & -3+\frac{h}{2} \end{bmatrix}$ The system is consistent if and only if $-3+\frac{h}{2} = 0$, that is, if and only if $h = 6$.

23. **a.** True. See the remarks following the box titled *Elementary Row Operations*.

b. False. A 5 × 6 matrix has five rows.

c. False. The description applies to a single solution. The solution *set* consists of all possible solutions. Only in special cases does the solution set consist of exactly one solution. Mark a statement True only if the statement is *always* true.

d. True. See the box before Example 2.

24. **a.** False. The definition of *row equivalent* requires that there exist a sequence of row operations that transforms one matrix into the other.

b. True. See the box preceding the subsection titled *Existence and Uniqueness Questions*.

c. False. The definition of *equivalent systems* is in the second paragraph after equation (2).

d. True. By definition, a consistent system has *at least one* solution.

25. $\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & k+2g \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & k+2g+h \end{bmatrix}$

Let b denote the number $k + 2g + h$. Then the third equation represented by the augmented matrix above is $0 = b$. This equation is possible if and only if b is zero. So the original system has a solution if and only if $k + 2g + h = 0$.

26. Row reduce the augmented matrix for the given system:

$$\begin{bmatrix} 2 & 4 & f \\ c & d & g \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & f/2 \\ c & d & g \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & f/2 \\ 0 & d-2c & g-c(f/2) \end{bmatrix}$$

This shows that $d - 2c$ must be nonzero, since f and g are arbitary. Otherwise, for some choices of f and g the second row would correspond to an equation of the form $0 = b$, where b is nonzero. Thus $d \neq 2c$.

27. Row reduce the augmented matrix for the given system. Scale the first row by $1/a$, which is possible since a is nonzero. Then replace R2 by R2 + $(-c)$R1.

$$\begin{bmatrix} a & b & f \\ c & d & g \end{bmatrix} \sim \begin{bmatrix} 1 & b/a & f/a \\ c & d & g \end{bmatrix} \sim \begin{bmatrix} 1 & b/a & f/a \\ 0 & d-c(b/a) & g-c(f/a) \end{bmatrix}$$

The quantity $d - c(b/a)$ must be nonzero, in order for the system to be consistent when the quantity $g - c(f/a)$ is nonzero (which can certainly happen). The condition that $d - c(b/a) \neq 0$ can also be written as $ad - bc \neq 0$, or $ad \neq bc$.

28. A basic principle of this section is that row operations do not affect the solution set of a linear system. Begin with a simple augmented matrix for which the solution is obviously (3, –2, –1), and then perform any elementary row operations to produce other augmented matrices. Here are three examples. The fact that they are all row equivalent proves that they all have the solution set (3, –2, –1).

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 1 & 0 & 4 \\ 2 & 0 & 1 & 5 \end{bmatrix}$$

29. Swap R1 and R3; swap R1 and R3.

30. Multiply R3 by –1/5; multiply R3 by –5.

31. Replace R3 by R3 + (–4)R1; replace R3 by R3 + (4)R1.

32. Replace R3 by R3 + (–4)R2; replace R3 by R3 + (4)R2.

33. The first equation was given. The others are:

$$T_2 = (T_1 + 20 + 40 + T_3)/4, \quad \text{or} \quad 4T_2 - T_1 - T_3 = 60$$
$$T_3 = (T_4 + T_2 + 40 + 30)/4, \quad \text{or} \quad 4T_3 - T_4 - T_2 = 70$$
$$T_4 = (10 + T_1 + T_3 + 30)/4, \quad \text{or} \quad 4T_4 - T_1 - T_3 = 40$$