Most important ideas:

- There's not a lot new today, just remembering several ideas from chapter 6 and putting them all together.
- Properties of a symmetric matrix, including its orthogonal diagonalization.
- Note: the spectrum refers to the entire range of frequencies (e.g. of color, sound, etc.)

First, another way (a little strange, but sometimes useful) to view matrix multiplication—see **Theorem 10 on page 119**.

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}\begin{bmatrix} 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix}\begin{bmatrix} 7 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 15 & 18 \end{bmatrix} + \begin{bmatrix} 14 & 16 \\ 28 & 32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}.$$

Second, recall that if an $n \times n$ matrix $P = [\,\vec{u}_1\, \vec{u}_2 \cdots \vec{u}_n\,]$ is orthogonal, that is, its columns are **orthonormal**, that is,

$$\vec{u}_i \cdot \vec{u}_j = \vec{u}_i^T \vec{u}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

then:

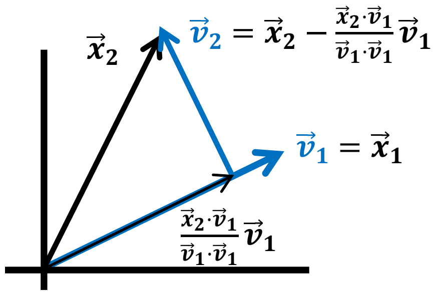
$$P^T P = \begin{bmatrix} \vec{u}_1^T \\ \vec{u}_2^T \\ \vdots \\ \vec{u}_n^T \end{bmatrix}\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix} = \begin{bmatrix} \vec{u}_1^T\vec{u}_1 & \vec{u}_1^T\vec{u}_2 & \cdots & \vec{u}_1^T\vec{u}_n \\ \vec{u}_2^T\vec{u}_1 & \vec{u}_2^T\vec{u}_2 & \cdots & \vec{u}_2^T\vec{u}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{u}_n^T\vec{u}_1 & \vec{u}_n^T\vec{u}_2 & \cdots & \vec{u}_n^T\vec{u}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

and remember that this also means that **the inverse of P is simply P^T, which is very useful.**

Third, recall the Gram-Schmidt Process (for two vectors): given $\vec{x}_1$ and $\vec{x}_2$, let

$$\vec{v}_1 = \vec{x}_1$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$$



Then $span\{\vec{v}_1, \vec{v}_2\} = span\{\vec{x}_1, \vec{x}_2\}$ but with $\vec{v}_1 \perp \vec{v}_2$:

$$\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \left(\vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1\right)$$

$$= \vec{v}_1 \cdot \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 \cdot \vec{v}_1$$

$$= 0$$

Fourth, recall that if for $n \times n$ A we have $A\vec{v}_1 = \lambda_1\vec{v}_1,\ A\vec{v}_2 = \lambda_2\vec{v}_2,\ \dots,\ A\vec{v}_n = \lambda_n\vec{v}_n$, then

$$A\,[\,\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n] = [\,A\vec{v}_1\ A\vec{v}_2\ \cdots\ A\vec{v}_n] = [\,\lambda_1\vec{v}_1\ \lambda_2\vec{v}_2\ \cdots\ \lambda_n\vec{v}_n] = [\,\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n]\begin{bmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{bmatrix}.$$

That is, where $P = [\,\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n]$ and $D = \begin{bmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{bmatrix}$, we have $AP = PD$.

and if the columns of P are linearly independent, then $\boldsymbol{P^{-1}}$ **exists** and $\boldsymbol{A = PDP^{-1}}$, that is, A is diagonalizable.

Fifth: if two eigenvectors share the same eigenvalue $A\vec{v}_1 = \lambda\vec{v}_1,\ A\vec{v}_2 = \lambda\vec{v}_2$ then

$$A(c_1\vec{v}_1 + c_2\vec{v}_2) = A(c_2\vec{v}_1) + A(c_2\vec{v}_2) = c_1A\vec{v}_1 + c_2A\vec{v}_2 = c_1\lambda\vec{v}_1 + c_2\lambda\vec{v}_2 = \lambda(c_1\vec{v}_1 + c_2\vec{v}_2)$$

so any linear combination of those two eigenvectors is also an eigenvector. (Of course the same sort of thing is true if there are more than two eigenvectors with the same eigenvalue.)

Let's put it all together in the following example.

Consider $A = \begin{bmatrix}4 & 1 & 3 & 1\\ 1 & 4 & 1 & 3\\ 3 & 1 & 4 & 1\\ 1 & 3 & 1 & 4\end{bmatrix}$, which is symmetric (i.e. $A^T = A$), and has eigenvalues and eigenvectors

$$\lambda_1 = 9 \qquad \lambda_2 = 5 \qquad \lambda_3 = \lambda_4 = 1$$

$$\vec{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}\ \vec{v}_4 = \begin{bmatrix}3\\2\\-3\\-2\end{bmatrix}$$

So $A = PDP^{-1}$ where

$$P = \begin{bmatrix}1 & 1 & 1 & 3\\ 1 & -1 & 1 & 2\\ 1 & 1 & -1 & -3\\ 1 & -1 & -1 & -2\end{bmatrix} \text{ and } D = \begin{bmatrix}9 & & & \\ & 5 & & \\ & & 1 & \\ & & & 1\end{bmatrix}.$$

You can check that $A = PDP^{-1}$. Notice that all of the eigenvectors are all mutually orthogonal, except for $\vec{v}_3$ and $\vec{v}_4$, which correspond to the same eigenvalue of 1. See Theorem 1 on page 395 and Theorem 2 on page 396.

Recall that if you have two or more vectors corresponding to the same eigenvalue, then any linear combination of those eigenvectors is also an eigenvector corresponding to the same eigenvalue. **Could we come up with two other eigenvectors corresponding to eigenvalue $\boldsymbol{\lambda_3 = \lambda_4 = 1}$ that are orthogonal to each other ("mutually orthogonal")? Yes, using the Gram-Schmidt Process. Flex.**

In the above example:

First, keep eigenvector $\vec{v}_3$.

Second, create a new eigenvector $\vec{v}_4$ still corresponding to $\lambda = 1$ but that is orthogonal to $\vec{v}_3$:

New $\boldsymbol{\vec{v}_4 = \vec{v}_4 - \frac{\vec{v}_4 \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3}\vec{v}_3 = \begin{bmatrix} 3 \\ 2 \\ -3 \\ -2 \end{bmatrix} - \frac{10}{4}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}}$.

We can actually use any multiple of an eigenvector, so let's multiply what we just found by 2 to avoid fractions. You can check that $A\vec{v}_4 = 1\vec{v}_4$ for $\vec{v}_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ and that this $\vec{v}_4$ is orthogonal to $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$.

So our new P is $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$ with the same $D = \begin{bmatrix} 9 & & & \\ & 5 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ as before.

(Again you can check that $A = PDP^{-1}$.)

We could take this one step further and normalize (make size 1) each column of P by simply dividing each column by its current size to get

$$P = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$$

So $A = PDP^{-1} = PDP^T$ since P is an orthogonal matrix. **So $\boldsymbol{A}$ is <u>orthogonally diagonalizable</u>.** (That would be a good name for a band, or at least a song.) Again see Theorem 2 on page 396.

Theorem 3 on page 397 summarizes a lot of this. Notice property (b) which guarantees that we will have a full set of eigenvectors.

So we see that symmetric matrices are really handy. Luckily many matrices that arise in real life problems are symmetric.

Last thought. Recall: if A is orthogonally diagonalizable, then

$$\begin{aligned} A &= PDP^T \\ &= [\vec{u}_1 \ \cdots \ \vec{u}_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{bmatrix} \\ &= [\lambda_1 \vec{u}_1 \ \cdots \ \lambda_n \vec{u}_n] \begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{bmatrix} \\ &= \lambda_1 \vec{u}_1 \vec{u}_1^T + \cdots + \lambda_n \vec{u}_n \vec{u}_n^T . \end{aligned}$$

Reminder: if $\vec{u}_i$ is $n \times 1$, then each $\vec{u}_i \vec{u}_i^T$ is an $n \times n$ matrix.

This is called the spectral decomposition of A. It sort of decomposes the matrix A into its various "frequencies," similar to how a Fourier Series (in Section 6.8) decomposes a function or signal into its various frequencies.

Example from above: $A = \lambda_1 \vec{u}_1 \vec{u}_1^T + \lambda_2 \vec{u}_2 \vec{u}_2^T + \lambda_3 \vec{u}_3 \vec{u}_3^T + \lambda_4 \vec{u}_4 \vec{u}_4^T$ where

$$\lambda_1 \vec{u}_1 \vec{u}_1^T = 9 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} [1/2 \quad 1/2 \quad 1/2 \quad 1/2\,] = \frac{9}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\lambda_2 \vec{u}_2 \vec{u}_2^T = 5 \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} [1/2 \quad -1/2 \quad 1/2 \quad -1/2\,] = \frac{5}{4} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

$$\lambda_3 \vec{u}_3 \vec{u}_3^T = 1 \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} [1/2 \quad 1/2 \quad -1/2 \quad -1/2\,] = \frac{1}{4} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

$$\lambda_4 \vec{u}_4 \vec{u}_4^T = 1 \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} [1/2 \quad -1/2 \quad -1/2 \quad 1/2\,] = \frac{1}{4} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Notice the "frequencies" of each of the above four matrices. Each eigenvalue λ_i tells us how much of $\vec{u}_i \vec{u}_i^T$ there is in A just like the Fourier coefficients a_k and b_k tell us how much of $\cos kt$ and $\sin kt$ there is in a particular function.