Most important ideas:

- Fourier Series: building a non-trigonometric function out of *cosines* and *sines*.
- Note: we are not covering weighted least squares, but you should at least spend five or ten minutes looking at the basic idea of it in the book.

First a bit of review of functions and Calculus.

If a function $f(t)$ is "odd" then $f(-t) = -f(t)$. Example: $\sin t$ is odd since: $\sin(-t) = -\sin t$.

If a function $f(t)$ is "even" then $f(-t) = f(t)$. Example: $\cos t$ is even since: $\cos(-t) = \cos t$.

Why are the terms "odd" and "even" used? Because it you look at the Taylor Series of a function, those functions that are odd contain only odd powers of x (or of t) and those that are even contain only even powers of x (or of t).

Example: The Taylor Series for $\sin t$ and $\cos t$.

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots \qquad \cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots$$

Notice that $sine$ has only odd powers of t, and $cosine$ has only even powers of t.
Most functions' Taylor Series include both odd and even powers of t, for example,

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$$

and thus is neither even nor odd.

Recall that we showed in Handout 6.7 that the collection of *cosines* and *sines*

$$\{\, 1, \cos t, \cos 2t, \ldots, \cos kt, \ldots, \\ \sin t, \sin 2t, \ldots, \sin kt, \ldots \,\}$$

is orthogonal under the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t) g(t)\, dt$$

Note: the book defines this integral to be from 0 to 2π rather from $-\pi$ to π, as seen on page 387. While this is also common, more mathematicians use (because it is more useful, in my opinion) the inner product with the integral from $-\pi$ to π, as I am doing.

Reminder: In Handout 6.7 we also saw that with this inner product

$$\langle 1, 1\rangle = 2\pi \quad \text{and} \quad \langle \cos kt, \cos kt\rangle = \pi \quad \text{and} \quad \langle \sin kt, \sin kt\rangle = \pi\,.$$

For $k = 1, 2, 3, \ldots$ we want to build any continuous function with this collection of *cosines* and *sines*. We know the *cosines* and *sines* are linearly independent since they are:

But do they span the entire collection of functions, that is, can we really build any function using *cosines* and *sines*? It turns out that yes we can. (We don't prove this here.) So how do we build a given function out of the *cosines* and *sines*?

Recall that if $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ is an orthogonal basis for W, then the projection of $\vec{y}$ onto W (that is, how we build $\vec{y}$ using the vectors in W) is

$$\vec{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 + \cdots + \frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\vec{u}_p$$

More generally (if we are dealing with inner products rather than dot products), if $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ is an orthogonal basis for W, then

$$\vec{y} = \frac{\langle\vec{y},\vec{u}_1\rangle}{\langle\vec{u}_1,\vec{u}_1\rangle}\vec{u}_1 + \frac{\langle\vec{y},\vec{u}_2\rangle}{\langle\vec{u}_2,\vec{u}_2\rangle}\vec{u}_2 + \cdots + \frac{\langle\vec{y},\vec{u}_p\rangle}{\langle\vec{u}_p,\vec{u}_p\rangle}\vec{u}_p$$

Suppose the "vector" $\vec{y}$ that we are talking about is function $f(t)$. Then since

$$\{\, 1, \cos t, \cos 2t, \ldots, \cos kt, \ldots,$$
$$\sin t, \sin 2t, \ldots, \sin kt, \ldots \,\}$$

is an orthogonal basis for the set of all continuous functions we have

$$f(t) = \frac{\langle f(t),1\rangle}{\langle 1,1\rangle}1 + \frac{\langle f(t),\cos t\rangle}{\langle \cos t,\cos t\rangle}\cos t + \frac{\langle f(t),\cos 2t\rangle}{\langle \cos 2t,\cos 2t\rangle}\cos 2t + \cdots + \frac{\langle f(t),\cos kt\rangle}{\langle \cos kt,\cos kt\rangle}\cos kt + \cdots$$

$$+ \frac{\langle f(t),\sin t\rangle}{\langle \sin t,\sin t\rangle}\sin t + \frac{\langle f(t),\sin 2t\rangle}{\langle \sin 2t,\sin 2t\rangle}\sin 2t + \cdots + \frac{\langle f(t),\sin kt\rangle}{\langle \sin kt,\sin kt\rangle}\sin kt + \cdots$$

We are projecting $f(t)$ onto the collection of *sine* and *cosine* functions; that is we are building $f(t)$ using the *sine* and *cosine* functions.

In general, as given above, the Fourier Series of $f(t)$ is

$$f(t) = a_0 + a_1 \cos t + a_2 \cos 2t + \cdots + a_k \cos kt + \cdots$$

$$b_1 \sin t + b_2 \sin 2t + \cdots + b_k \sin kt + \cdots$$

$$= a_0 + \sum_{k=0}^{\infty} a_k \cos kt + b_k \sin kt$$

where

$$a_0 = \frac{\langle f(t),1\rangle}{\langle 1,1\rangle} = \frac{\langle f(t),1\rangle}{2\pi} = \frac{\int_{-\pi}^{\pi} f(t)\,dt}{2\pi}.$$

and for $k = 1, 2, \ldots$

$$a_k = \frac{\langle f(t),\cos kt\rangle}{\langle \cos kt,\cos kt\rangle} = \frac{\langle f(t),\cos kt\rangle}{\pi} = \frac{\int_{-\pi}^{\pi} f(t)\cos kt\;dt}{\pi}.$$

$$b_k = \frac{\langle f(t),\sin kt\rangle}{\langle \sin kt,\sin kt\rangle} = \frac{\langle f(t),\sin kt\rangle}{\pi} = \frac{\int_{-\pi}^{\pi} f(t)\sin kt\;dt}{\pi}.$$

Example: Suppose $f(t) = t$.

Then

$$a_0 = \frac{\int_{-\pi}^{\pi} t \ dt}{2\pi} = \frac{1}{2\pi} t^2 \Big|_{-\pi}^{\pi} = \frac{1}{2\pi}(\pi^2 - (-\pi)^2) = 0$$

$$a_1 = \frac{\int_{-\pi}^{\pi} t \ \cos t \ \ dt}{\pi} = \frac{1}{\pi}(t \sin t + \cos t)\Big|_{-\pi}^{\pi} = 0$$

And in general (to save time, so that we don't have to individually find a_2, a_3, a_4, etc.)

$$a_k = \frac{\int_{-\pi}^{\pi} t \ \cos kt \ \ dt}{\pi} = \frac{1}{\pi}\left(\frac{t \sin kt}{k} + \frac{\cos kt}{k^2}\right)\Big|_{-\pi}^{\pi} = 0$$

since $\sin k\pi = 0$ for any integer k and $\cos k(-\pi) = \cos k\pi$.

So there are no *cosine* terms part of the Fourier Series for the function t.

Let's again save time and find b_k in general, rather than one term $b_1, b_2, b_3, \ldots$ at a time:

$$\begin{aligned} b_k &= \frac{\int_{-\pi}^{\pi} t \sin kt \ \ dt}{\pi} = \frac{1}{\pi}\left(\frac{\sin kt}{k^2} - \frac{t \cos kt}{k}\right)\Big|_{-\pi}^{\pi} \\ &= \frac{1}{\pi}\left[\left(\frac{\sin k\pi}{k^2} - \frac{\pi \cos k\pi}{k}\right) - \left(\frac{\sin(-k\pi)}{k^2} - \frac{(-\pi)\cos(-k\pi)}{k}\right)\right] \\ &= -\frac{\pi}{\pi}\left[\frac{\cos(k\pi)}{k} + \frac{\cos(-k\pi)}{k}\right] \end{aligned}$$

since $\sin k\pi = 0$ and $\sin(-k\pi) = 0$ for any integer k

$$= -\frac{2}{k}[\cos k\pi] = \frac{2}{k}(-1)^{k+1}$$

since $\cos(-k\pi) = \cos k\pi$ and $\cos k\pi = 1$ for k odd and $\cos k\pi = -1$ for k even.

So

$$t = \frac{2}{1}\sin t - \frac{2}{2}\sin 2t + \frac{2}{3}\sin 3t - \frac{2}{4}\sin 4t + \frac{2}{5}\sin 5t - \cdots$$

$$= 2\left(\sin t - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \frac{\sin 4t}{4} + \frac{\sin 5t}{5} - \cdots\right)$$

Are you kidding me? So function t is an infinite sum of *sine* terms? **It turns out that Fourier Series have all sorts of applications, including in working with audio signals and images.**

If $f(t)$ is an odd function, e.g. $f(t) = t$ then its Fourier Series contains only *sine* terms and if $f(t)$ is an even function, e.g. $f(t) = t^2$ then its Fourier Series contains only *cosine* terms. Most functions, e.g. $f(t) = e^t$, are a mix of odd and even and thus have Fourier Series which contain both *cosines* and *sines*. The Fourier Series decomposes a function (or an audio signal) into its various frequencies: each coefficient a_k or b_k is the magnitude of that frequency.