Most important ideas:

- Inner products and inner product spaces
- The norm of a function; orthogonality of functions

Recall that dot products can be used to:

1. Find the size of a vector: $\|\vec{\boldsymbol{u}}\| = \sqrt{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}} = \sqrt{\vec{\boldsymbol{u}}^T\vec{\boldsymbol{u}}}$.
2. They allow us to define/recognize when two vectors are orthogonal: $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = \boldsymbol{0}$

Orthogonality is useful because:

1. If $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ is an orthogonal basis for W, then for any $\vec{y} \in W$, then we have a nice formula for how to build $\vec{y}$ out of $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p$:
$$\vec{\boldsymbol{y}} = \frac{\vec{\boldsymbol{y}} \cdot \vec{\boldsymbol{u}}_1}{\vec{\boldsymbol{u}}_1 \cdot \vec{\boldsymbol{u}}_1}\vec{\boldsymbol{u}}_1 + \frac{\vec{\boldsymbol{y}} \cdot \vec{\boldsymbol{u}}_2}{\vec{\boldsymbol{u}}_2 \cdot \vec{\boldsymbol{u}}_2}\vec{\boldsymbol{u}}_2 + \cdots + \frac{\vec{\boldsymbol{y}} \cdot \vec{\boldsymbol{u}}_p}{\vec{\boldsymbol{u}}_p \cdot \vec{\boldsymbol{u}}_p}\vec{\boldsymbol{u}}_p$$
Recall that this formula is only true if the vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p$ are orthogonal.

2. Orthogonality also leads to the Gram-Schmidt Process.

Sometimes it is very useful to build one function out of a collection of other functions. For example,
$$x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right)$$

Try plotting $2(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4})$ to see how it approximates or sort of "fits" the function x. Without getting into too much detail right now about building of x out of *sine* functions, for now I'll just say that we need the idea of orthogonality for functions to figure out how to build x out of *sines*.

Recall that dot products allow us to determine whether two vectors are orthogonal: their dot product $= 0$ means they are orthogonal. Again, we need a way to determine whether two functions are orthogonal (don't worry about what this looks like—it doesn't look like anything!)

An inner product is a more general idea of a dot product. Recall the theorem about dot products on page 331, as well as the idea of size/length/norm on page 331 and orthogonality on page 334. Compare these to the definition of an inner product on page 376 and size/length/norm and orthogonality on page 377. (Why is the one set of rules a theorem and the other set of rules a definition?) Spend a few minutes looking at these things in the book.

The dot product $\vec{u} \cdot \vec{v} = \vec{u}^T\vec{v}$ is just one version of an inner product, a version used for vectors.

Recall that for dot product of vectors: $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$

Similarly, for inner products: $\|\vec{\boldsymbol{u}}\| = \sqrt{\langle \vec{\boldsymbol{u}}, \vec{\boldsymbol{u}} \rangle}$

Example 1: Given functions $f(t)$ and $g(t)$, define

$$\langle f, g \rangle = \int_{-1}^{1} f(t) g(t)\, dt$$

There are all sorts of ways to define the inner product between two functions (or two vectors or two matrices or two digital images or two of anything from a vector space). The inner product defined simply has to satisfy the properties listed on page 376. (Does the inner product just defined satisfy the properties on page 376? We could show that it does.)

Suppose $f(t) = t$, $g(t) = t^2$. Then

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_{-1}^{1} t \cdot t\, dt} = \sqrt{\int_{-1}^{1} t^2\, dt} = \sqrt{2/3}$$

$$\|g\| = \sqrt{\langle g, g \rangle} = \sqrt{\int_{-1}^{1} t^2 \cdot t^2\, dt} = \sqrt{\int_{-1}^{1} t^4\, dt} = \sqrt{2/5}$$

$$\langle f, g \rangle = \int_{-1}^{1} t \cdot t^2\, dt = \int_{-1}^{1} t^3\, dt = 0$$

Important: $\langle f, g \rangle = 0$ means that functions $f(t) = t$ and $g(t) = t^2$ are orthogonal, under the inner product just defined above.

There is no way to visualize the idea of two functions being orthogonal. This doesn't matter at all, as what we what we want to do with functions being orthogonal, like to project one function onto an orthogonal set of functions (what does it mean to project a vector onto a set of vectors?) doesn't require us to visualize anything.

Side note: recall the inequalities that we saw in 6.1 for dot products:

$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$, that is, the size of the product is ≤ the product of the sizes

$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$, that is, the size of the sum is ≤ the sum of the sizes

These are true with inner products:

$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\|\|\vec{v}\|$, that is, the size of the product is ≤ the product of the sizes

$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$, that is, the size of the sum is ≤ the sum of the sizes

Side note: The Wronskian of 2 or more functions is a way to determining whether or not those functions are linearly independent. This is something you will use more in Math 340, Differential Equations.

Example 2: $f(t) = 4 + t,\ f'(t) = 1$
$g(t) = 5 - 4t^2,\ g'(t) = -8t$

So the Wronskian of f and g is $\begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} 4+t & 5-4t^2 \\ 1 & -8t \end{vmatrix} = -4t^2 - 32t - 5 \neq 0$, which means that functions f and g are linearly independent.

Example 3:

$$\begin{vmatrix} 4+t & 5-4t^2 & 8t^2+3t+2 \\ 1 & -8t & 16t+3 \\ 0 & -8 & 16 \end{vmatrix} = (4+t)\begin{vmatrix} -8t & 16t+3 \\ -8 & 15 \end{vmatrix} - 1\begin{vmatrix} 5-4t^2 & 8t^2+3t+2 \\ -8 & 16 \end{vmatrix} + 0\begin{vmatrix} 5-4t^2 & 8t^2+3t+2 \\ -8t & 16t+3 \end{vmatrix} = 0$$

So the collection of functions $4 + t,\ 5 - 4t^2,\ 8t^2 + 3t + 2$ is linearly dependent.
(Notice that $8t^2 + 3t + 2 = 3(4 + t) - 2(5 - 4t^2)$.)

We can project one function onto another. Let's work Class Example 4 in class.

We can use the Gram-Schmidt Process on a collection of linearly independent functions to create an orthogonal set of functions which span (generate) the same set of functions. Let's work Class Example 5 in class.

Example 6: Consider the set of functions

$$\{\, 1, \cos t\,, \cos 2t\,, \dots, \cos kt\,, \dots\ ,\ \sin t\,, \sin 2t\,, \dots, \sin kt\,, \dots \,\}$$

where k is a positive integer, along with the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)g(t)\,dt\,.$$

It turns out that the above set of *cosine* and *sine* functions is orthogonal under this inner product. We want to build other (non-trig) functions with these trigonometric functions.

Note that $\cos 0t = \cos 0 = 1$ and $\sin 0t = \sin 0 = 0$, but we won't include 0 in the above set of *cosines* and *sines*, as including 0 wouldn't allow us to build anything else, and the set would not be linearly independent (any set of vectors with the zero vector is linearly independent).

In 6.8 we will build functions (as linear combinations, naturally) using the above *cosines* and *sines*. For now, let's simply show that these functions are orthogonal:

First, $$\langle 1, \cos kt\rangle = \int_{-\pi}^{\pi} 1\cdot \cos kt \ dt = \frac{\sin kt}{k}\bigg|_{-\pi}^{\pi} = \frac{1}{k}(\sin k\pi - \sin(-k\pi)) = 0.$$

since $\sin k\pi = \sin(-k\pi) = 0$ for any integer k, and

$$\langle 1, \sin kt\rangle = \int_{-\pi}^{\pi} 1\cdot \sin kt \ dt = -\frac{\cos kt}{k}\bigg|_{-\pi}^{\pi} = -\frac{1}{k}(\cos k\pi - \cos(-k\pi)) = 0.$$

since $\cos\theta = \cos(-\theta)$ for any θ.

Next, for $m \neq n$ (using Wolfram Alpha),

$$\langle \cos mt\,, \cos nt\rangle = \int_{-\pi}^{\pi} \cos mt \cdot \cos nt \ dt = \frac{m \sin mt \cos nt - n \cos mt \sin nt}{m^2 - n^2}\bigg|_{-\pi}^{\pi} = 0$$

since $\sin m\pi = 0$ and $\sin n\pi = 0$ where m and n are integers. We could do the same with the *sine* vs. *sine* functions and the *cosine* vs. *sine* functions, to then conclude that each function in

$$\{\, 1, \cos t\,, \cos 2t\,, \dots, \cos kt\,, \dots \ , \ \sin t\,, \sin 2t\,, \dots, \sin kt\,, \dots \,\}$$

is orthogonal to every other function.

Finally, also useful in Section 6.8 is the fact that

$$\boldsymbol{\langle 1, 1\rangle = \int_{-\pi}^{\pi} 1\cdot 1 \, dt = 2\pi.}$$

and using the fact that $\cos 2t = 2\cos^2 t - 1 \ \Rightarrow \ \cos^2 t = \frac{1+\cos 2t}{2}$

$$\boldsymbol{\langle \cos kt\,, \cos kt\rangle = \int_{-\pi}^{\pi} \cos^2 kt \ dt = \int_{-\pi}^{\pi} \frac{1+\cos 2t}{2} \ dt = \pi}$$

and using the fact that $\cos 2t = 1 - 2\sin^2 t \ \Rightarrow \ \sin^2 t = \frac{1-\cos 2t}{2}$

$$\boldsymbol{\langle \sin kt\,, \sin kt\rangle = \int_{-\pi}^{\pi} \sin^2 kt \ dt = \int_{-\pi}^{\pi} \frac{1-\cos 2t}{2} \ dt = \pi.}$$

Or you could be clever and use the fact that $\sin^2 kt = 1 - \cos^2 kt$ so that

$$\boldsymbol{\langle \sin kt\,, \sin kt\rangle = \int_{-\pi}^{\pi} \sin^2 kt \ dt = \int_{-\pi}^{\pi} 1 - \cos^2 kt \, dt = 2\pi - \pi = \pi.}$$