Most important ideas:

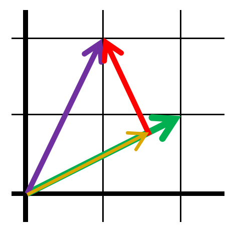
- Dot product (also called inner product); norm/length of a vector.
- Properties of dot products given in Theorem 1 on page 331.
- Orthogonality of two vectors: dot product is 0.
  Orthogonality of one vector to a vector space.
  Orthogonality of one vector space to another vector space: all vectors in the one vector space are orthogonal to all vectors in the other.

First, a little motivation about why we need the idea of two vectors being perpendicular in dimensions higher than R^3, dimensions in which we can't visualize things.

What is the point on the floor closest to a particular fire sprinkler on the ceiling?

The point on the floor that is directly below the fire sprinkler.

What is the point along $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ that is closest to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$?



The point looks like it is a bit before point (2,1) .
We'll later find that the exact point is (8/5, 4/5) .

Why does this point **(8/5, 4/5)** seem correct? **The vector (-3/5, 6/5) that points from (8/5, 4/5) to (2,1) seems *perpendicular* to the vector (2,1).**

Notice that $\begin{bmatrix} -3/5 \\ 6/5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \mathbf{0}$, which means the two vectors are indeed perpendicular.

So how could we find the point in R^4 along $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ that is closest to $\begin{bmatrix} -2 \\ 3 \\ 0 \\ 5 \end{bmatrix}$?

We can't draw a picture of this, so to find the desired vector, we need a way to describe/recognize when a vector in R^4 (or higher dimension) is "perpendicular" to another vector without a picture. **In general, we refer to "perpendicular" as orthogonal** .

Vectors $\vec{u}$ and $\vec{v}$ in R^2 or R^3 are perpendicular if: $\vec{u} \cdot \vec{v} = 0$.

$\vec{u} \cdot \vec{v}$ is called the dot product of $\vec{u}$ and $\vec{v}$. Also useful to us: $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$. See page 330.

In general, **two vectors $\vec{u}$ and $\vec{v}$ of any size** ($\vec{u}$ and $\vec{v}$ can be of any size, but they must be of the *same* size, e.g. both are from R^3 or both are from R^4, etc.) **are orthogonal if $\vec{u} \cdot \vec{v} = 0$. Notation: $\vec{u} \perp \vec{v}$ is the notation that $\vec{u}$ and $\vec{v}$ are orthogonal.** (We'll see later that we can also have orthogonal functions and other non-vectors.)

Example 1: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 3 \\ 0 \\ -1 \end{bmatrix} = (1)(-2) + (2)(3) + (3)(0) + (4)(-1) = 0,$ so the vectors are orthogonal.

What does this look like? I don't know—we can't visualize it—luckily we don't need to visualize any of this for the idea of orthogonality to be extremely useful to us.

The length/size/norm of a vector $\vec{v}$ is $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$.

Example 2: For $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $||\vec{u}|| = \sqrt{(\mathbf{2})(\mathbf{2}) + (\mathbf{1})(\mathbf{1})} = \sqrt{\mathbf{5}}$. **What does this look like?**

We don't need a diagram to compute the length of vector (and of course for vectors from R^4 and higher, we can't draw a diagram, even if we wanted to).

For $\vec{u} = \begin{bmatrix} 14 \\ 7 \end{bmatrix}$, $||\vec{u}|| = \sqrt{(14)(14) + (7)(7)} = \sqrt{245} = 7\sqrt{5}$

For $\vec{u} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$, $||\vec{u}|| = \sqrt{(-2)(-2) + (-1)(-1)} = \sqrt{5}$

For $\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $||\vec{u}|| = \sqrt{(1)(1) + (-2)(-2) + (3)(3)} = \sqrt{14}$

For $\vec{u} = \begin{bmatrix} 2 \\ -5 \\ 0 \\ 3 \end{bmatrix}$, $||\vec{u}|| = \sqrt{(2)(2) + (-5)(-5) + (0)(0) + (3)(3)} = \sqrt{38}$

Property (d) in Theorem 1 tells us that the size/length of a vector is always positive, unless the vector is the zero vector, in which case its size is 0. Also: $||c\vec{v}|| = |c|||\vec{v}||$, as we saw above.

Note: for any vector $\vec{v}$, $\frac{\vec{v}}{||\vec{v}||}$ **is the <u>unit vector</u> in the same direction as $\vec{v}$ but with *unit* length (length 1).**

Example 3: $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ so $\frac{\vec{v}}{||\vec{v}||} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$, which has size $\sqrt{\left(\frac{2}{\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}}\right) + (\frac{1}{\sqrt{5}})(\frac{1}{\sqrt{5}})} = \sqrt{\frac{4}{5} + \frac{1}{5}} = \mathbf{1}.$

$\vec{v} = \begin{bmatrix} 2/4 \\ 1/4 \end{bmatrix}$ so $\frac{\vec{v}}{||\vec{v}||} = \frac{1}{\sqrt{5/16}}\begin{bmatrix} 2/4 \\ 1/4 \end{bmatrix} = \frac{4}{\sqrt{5}}\begin{bmatrix} 2/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$.

In general, the size of a vector $\frac{\vec{v}}{||\vec{v}||}$ that has been normalized (divided by its own length/size) is

$$\left\|\frac{\vec{v}}{||\vec{v}||}\right\| = \sqrt{\frac{\vec{v}}{||\vec{v}||} \cdot \frac{\vec{v}}{||\vec{v}||}} = \sqrt{\left(\frac{1}{||\vec{v}||}\right)\left(\frac{1}{||\vec{v}||}\right)(\vec{v} \cdot \vec{v})} = \frac{1}{||\vec{v}||}\sqrt{\vec{v} \cdot \vec{v}} = \frac{1}{||\vec{v}||}||\vec{v}|| = \mathbf{1}.$$

(Remember that $||\vec{v}||$ is a simply a number.)

The distance between two vectors (points) is $dist(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}||$. See Figure 4 on page 333.

On page 333: $[dist(u, v)]^2 = ||\vec{u} - \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2 - 2\,\vec{u} \cdot \vec{v}$.

$[dist(u, -v)]^2 = ||\vec{u} + \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2 + 2\,\vec{u} \cdot \vec{v}$.

If $\vec{u} \perp \vec{v}$, then $\vec{u} \cdot \vec{v} = 0$, and $||\vec{u} + \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2$. This is the **Pythagorean Theorem**. See page 334. This is true in any dimension, including dimensions we can't visualize (R^4 and higher). In R^2 and R^3, $\vec{u} \cdot \vec{v} = ||\vec{u}||\,||\vec{v}|| \cos\theta$ where θ is angle between $\vec{u}$ and $\vec{v}$. See Figure on page 335. We will prove two simple but extremely useful properties in class. These are true in any dimension:

$||\vec{u} + \vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$ The size of the sum $\leq$ the sum of the sizes. This is the Triangle Inequality.

$|\vec{u} \cdot \vec{v}| \leq ||\vec{u}||||\vec{v}||$ The size of the (dot) product $\leq$ the product of the sizes.

Next, **given a vector space W (a collection of vectors) from R^n, there is another vector space (another collection of vectors), also from R^n, that is orthogonal to W.**

Example 4: $W = span\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right\}$. So W is a subspace of vectors in R^3, since W is the collection of all vectors than are linear combinations) of $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}4\\5\\6\end{bmatrix}$, which come from R^3.

What is the dimension of W? **2, since there are 2 linearly independent vectors in the basis for W.**

What do you think W looks like? **W would be a 2-dimensional object that exists (sometimes we say "lives") in R^3. So W is a plane which passes through the origin.**

So let's find $W^\perp$, the collection of all vectors that are orthogonal $\perp$ to all of the vectors in W. **Vector space $W^\perp$ is called the <u>orthogonal complement</u> of W.** (One meaning of the word "complement" is "the part left over" or "the other part". For example, the complement of/to *the ladies* in class would be *the men* in class.) A vector is orthogonal to all vectors in W if and only if it is orthogonal to each vector in the basis of W. Let's show this in class.

So $W^\perp$ consists of all vectors $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ for which

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} = 1x_1 + 2x_2 + 3x_3 = 0 \quad \text{and} \quad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \cdot \begin{bmatrix}4\\5\\6\end{bmatrix} = 4x_1 + 5x_2 + 6x_3 = 0$$

That is,

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

Then

$$\left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\4 & 5 & 6 & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & -1 & 0\\0 & 1 & 2 & 0\end{array}\right] \Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_3\\-2x_3\\x_3\end{bmatrix} = x_3\begin{bmatrix}1\\-2\\1\end{bmatrix}.$$

So $W^\perp$ consists of all multiples of the vector $\begin{bmatrix}1\\-2\\1\end{bmatrix}$, a 1-dimensional vector space (a line that pass through the origin).

You can verify that $\begin{bmatrix}1\\-2\\1\end{bmatrix} \perp$ both $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}4\\5\\6\end{bmatrix}$.

If W is a subspace of R^n, then $\dim W + \dim W^\perp = n$. In example above, we had $2 + 1 = 3$.

See Book Figure 7 on page 334 for a picture of this.

Example 5: Let $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$, and let $W = Col\ A = span\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}\right\}$.

Since by definition $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}\right\}$ generates/spans W, **then $W^{\perp}$ consists of all of the vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ that are orthogonal to these three vectors**:

$$\left.\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} &= 1x_1 + 2x_2 + 3x_3 = 0 \\ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} &= 4x_1 + 5x_2 + 6x_3 = 0 \\ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} &= 7x_1 + 8x_2 + 9x_3 = 0 \end{aligned}\right\}$$

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ leads to

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So $W^{\perp}$ consists of all multiples of $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Notice that $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \perp$ each column of A.

We had already seen in Example 4 that $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, so any vector built out of them, including $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, will also be orthogonal to $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

In the above example,

$$Col\ A = span\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}\right\} = span\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right\}.$$

Note: In the above example, the vectors $\vec{x}$ that are $\perp W$ are the same vectors in the null space of A^T. That is,

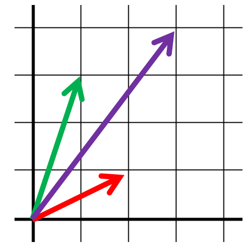
$$\boldsymbol{(Col\ A)^{\perp} = Nul\ A^T,}$$

and similarly (replacing A^T with A and A with A^T) we have

$$\boldsymbol{(Col\ A^T)^{\perp} = (Row\ A)^{\perp} = Nul\ A.}$$

This is Theorem 3 on page 335. This seems like kind of a strange result, but it's pretty straightforward if you think (slowly) about it.

Compare the Law of Cosines (Google it) to the bottom half of Book page 335.



Two cases: if $\theta < 90°$ or if $\theta > 90°$

(if $\theta = 90°$, we have the Pythagorean Theorem)

Example 6: $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

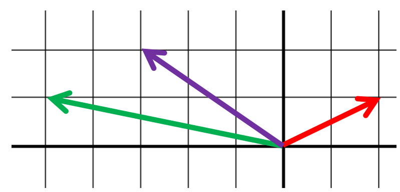
Note that $\|\vec{u}\| = \sqrt{5} \approx 2.24$ and $\|\vec{v}\| = \sqrt{10} \approx 3.16$.

$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\,\vec{u} \cdot \vec{v} = 5 + 10 + 2(5) = 25$, so $\|\vec{u} + \vec{v}\|^2 = 5$.

(Compare this to using the Pythagorean Theorem to see that the size of vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is 5.)

$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{5}{(\sqrt{5})(\sqrt{10})} = \frac{5}{\sqrt{50}} = \frac{\sqrt{2}}{2} \Rightarrow \theta = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45°$ is the angle between $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

In general, if $\theta < 90°$, then $\cos\theta > 0$ which means $\vec{u} \cdot \vec{v} > 0$, so $\|\vec{u} + \vec{v}\|^2 > \|\vec{u}\|^2 + \|\vec{v}\|^2$.
(Notice that for the above example we have $5^2 > \left(\sqrt{5}\right)^2 + \left(\sqrt{10}\right)^2$.)



Example 7: $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$.

Note that $\|\vec{u}\| = \sqrt{5} \approx 2.24$ and $\|\vec{v}\| = \sqrt{26} \approx 5.10$.

$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\,\vec{u} \cdot \vec{v} = 5 + 26 + 2(-9) = 13$, so $\|\vec{u} + \vec{v}\|^2 = \sqrt{13}$.

(Compare this to using the Pythagorean Theorem to see that the size of vector $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ is $\sqrt{13}$.)

$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = -\frac{9}{(\sqrt{5})(\sqrt{26})} = -\frac{9}{\sqrt{130}} \Rightarrow \theta = \cos^{-1}\left(-\frac{9}{\sqrt{130}}\right) \approx 142°$ is the angle between $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 1 \end{bmatrix}$.

In general, if $\theta > 90°$, then $\cos\theta < 0$ which means $\vec{u} \cdot \vec{v} < 0$, so $\|\vec{u} + \vec{v}\|^2 < \|\vec{u}\|^2 + \|\vec{v}\|^2$.
(Notice that for the above example we have $\left(\sqrt{13}\right)^2 < \left(\sqrt{5}\right)^2 + \left(\sqrt{26}\right)^2$.)

The formula $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\,\vec{u} \cdot \vec{v}$ is most useful for R^3 as a theoretical result.