Most important ideas:

- What happens if we repeatedly multiply a vector by a matrix? How to explain this in terms of the eigenvalues and eigenvectors of that matrix? (We've seen this already.)
- The Power Method and Inverse Power Method for estimating the largest and smallest eigenvalues and their corresponding eigenvectors.

First, if $A\vec{v} = \lambda\vec{v}$, then $(cA)\vec{v} = c(A\vec{v}) = c(\lambda\vec{v}) = (c\lambda)\vec{v}$. That is, cA has the same eigenvectors as A but with eigenvalues multiplied by c.

Example: $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 14 & 21 \\ 7 & 28 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 35 \\ 35 \end{bmatrix} = 35\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Next, recall that **if $A\vec{v} = \lambda\vec{v}$ then** $A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda(A\vec{v}) = \lambda(\lambda\vec{v}) = \lambda^2\vec{v}$, and in general $\boldsymbol{A^k\vec{v} = \lambda^k\vec{v}}$ (including for $\lambda = -1$: $A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$).

Example: $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}^2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 25 \\ 25 \end{bmatrix} = 5^2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Example: Where $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 4/5 & -3/5 \\ -1/5 & 2/5 \end{bmatrix}$, we have $\begin{bmatrix} 4/5 & -3/5 \\ -1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 1/5 \end{bmatrix} = 1/5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Second, recall that if $n \times n$ matrix A has a complete (linearly independent) set of eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ which form a basis for R^n, then for any vector $\vec{x} \in R^n$,

$$\boldsymbol{\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n}$$

and

$$\boldsymbol{A^k\vec{x} = A^k(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) = A^k c_1\vec{v}_1 + A^k c_2\vec{v}_2 + \cdots + A^k c_n\vec{v}_n}$$
$$\boldsymbol{= \lambda_1^k\, c_1\vec{v}_1 + \lambda_2^k\, c_2\vec{v}_2 + \cdots + \lambda_n^k\, c_n\vec{v}_n}$$

Example: Suppose $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix},\ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & -.8 \end{bmatrix} \Rightarrow A = \begin{bmatrix} .15 & .60 & -.35 \\ .60 & .40 & .60 \\ -.35 & .60 & .15 \end{bmatrix}$$

Eigenvectors: $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Eigenvalues: 1, .5, $-.8$

We'll now use the fact that $A^k\vec{x} = \lambda_1^k\, c_1\vec{v}_1 + \lambda_2^k\, c_2\vec{v}_2 + \cdots + \lambda_n^k\, c_n\vec{v}_n$.

Suppose $\vec{x} = \begin{bmatrix}5\\2\\9\end{bmatrix} = 3\begin{bmatrix}1\\2\\1\end{bmatrix} + (-2)\begin{bmatrix}1\\0\\-1\end{bmatrix} + 4\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}3\\6\\3\end{bmatrix} + \begin{bmatrix}-2\\0\\2\end{bmatrix} + \begin{bmatrix}4\\-4\\4\end{bmatrix}$.

What happens if we repeatedly multiply $\vec{x}$ by A?

$$\boldsymbol{A\vec{x} = (1)\begin{bmatrix}3\\6\\3\end{bmatrix} + (.5)\begin{bmatrix}-2\\0\\2\end{bmatrix} + (-.8)\begin{bmatrix}4\\-4\\4\end{bmatrix} = \begin{bmatrix}-1.2\\9.2\\0.8\end{bmatrix}.}$$

$$\boldsymbol{A^2\vec{x} = (1)^2\begin{bmatrix}3\\6\\3\end{bmatrix} + (.5)^2\begin{bmatrix}-2\\0\\2\end{bmatrix} + (-.8)^2\begin{bmatrix}4\\-4\\4\end{bmatrix} = \begin{bmatrix}5.06\\3.44\\6.06\end{bmatrix}.}$$

$$\boldsymbol{A^3\vec{x} = (1)^3\begin{bmatrix}3\\6\\3\end{bmatrix} + (.5)^3\begin{bmatrix}-2\\0\\2\end{bmatrix} + (-.8)^3\begin{bmatrix}4\\-4\\4\end{bmatrix} = \begin{bmatrix}0.702\\8.048\\1.202\end{bmatrix}.}$$

$$\boldsymbol{A^{10}\vec{x} = (1)^{10}\begin{bmatrix}3\\6\\3\end{bmatrix} + (.5)^{10}\begin{bmatrix}-2\\0\\2\end{bmatrix} + (-.8)^{10}\begin{bmatrix}4\\-4\\4\end{bmatrix} = \begin{bmatrix}3.42754\\5.57050\\3.43145\end{bmatrix}.}$$

$$\boldsymbol{A^{100}\vec{x} = (1)^{100}\begin{bmatrix}3\\6\\3\end{bmatrix} + (.5)^{100}\begin{bmatrix}-2\\0\\2\end{bmatrix} + (-.8)^{100}\begin{bmatrix}4\\-4\\4\end{bmatrix} \approx \begin{bmatrix}3\\6\\3\end{bmatrix}.}$$

$$\boldsymbol{A^{\infty}\vec{x} = (1)^{\infty}\begin{bmatrix}3\\6\\3\end{bmatrix} + (.5)^{\infty}\begin{bmatrix}-2\\0\\2\end{bmatrix} + (-.8)^{\infty}\begin{bmatrix}4\\-4\\4\end{bmatrix} = \begin{bmatrix}3\\6\\3\end{bmatrix}.}$$

Observation: As we repeatedly multiply the vector $\vec{x}$ by A, the part of $\vec{x}$ that comes from the eigenvectors corresponding to the < 1 eigenvalues of 0.5 and -0.8 disappear (they are "destroyed"), while the part of $\vec{x}$ that comes from the eigenvector corresponding to the largest (the "dominant") eigenvalue 1 is what remains.

Next, suppose A were some multiple of the A we created above? Say $A = \begin{bmatrix}1.5 & 6.0 & -3.5\\6.0 & 4.0 & 6.0\\-3.5 & 6.0 & 1.5\end{bmatrix}$.

This A has the same eigenvectors as the A above but with eigenvalues that are 10 times the eigenvalues as A above, so eigenvalues of $10,\ 5,\ -8$.

For the same $\vec{x} = \begin{bmatrix} 5 \\ 2 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix}$ as above we have

$$\boldsymbol{A\vec{x} \;=\; (10)\begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} + (5)\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} + (-8)\begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} -12 \\ 92 \\ 8 \end{bmatrix}.}$$

$$\boldsymbol{A^2\vec{x} = (10)^2\begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} + (5)^2\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} + (-8)^2\begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} 506 \\ 344 \\ 606 \end{bmatrix}.}$$

$$\boldsymbol{A^{20}\vec{x} = (10)^{20}\begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} + (5)^{20}\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} + (-8)^{20}\begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix} \approx \begin{bmatrix} 3.05 \\ 5.95 \\ 3.05 \end{bmatrix} \times 10^{20}.}$$

And so on. We see that once again the term that is gradually starting to dominate the other two terms is the term (the eigenvector) that corresponds to the largest eigenvalue.

The resulting vector is starting to become large, really large—too large—so after each multiplication let's divide the new vector by the largest value in the vector. For $\vec{x}_0$ we'll again use $(5, 2, 9)$. Note: μ is a Greek m (perhaps standing for max?). The results:

k	0	1	2	10	20	30	100
$\vec{x}_k$	$\begin{bmatrix} 5 \\ 2 \\ 9 \end{bmatrix}$	$\begin{bmatrix} -0.1304 \\ 1.0000 \\ 0.0870 \end{bmatrix}$	$\begin{bmatrix} 0.8350 \\ 0.5677 \\ 1.0000 \end{bmatrix}$	$\begin{bmatrix} 0.6153 \\ 1.0000 \\ 0.6160 \end{bmatrix}$	$\begin{bmatrix} 0.5116 \\ 1.0000 \\ 0.5116 \end{bmatrix}$	$\begin{bmatrix} 0.5012 \\ 1.0000 \\ 0.5012 \end{bmatrix}$	$\begin{bmatrix} 0.5000 \\ 1.0000 \\ 0.5000 \end{bmatrix}$
$\vec{x}_{k+1} = A\vec{x}_k$	$\begin{bmatrix} -12 \\ 92 \\ 8 \end{bmatrix}$	$\begin{bmatrix} 5.5000 \\ 3.7391 \\ 6.5870 \end{bmatrix}$	$\begin{bmatrix} 1.1584 \\ 13.2805 \\ 1.9835 \end{bmatrix}$	$\begin{bmatrix} 4.7669 \\ 11.3878 \\ 4.7704 \end{bmatrix}$	$\begin{bmatrix} 4.9768 \\ 10.1394 \\ 4.9768 \end{bmatrix}$	$\begin{bmatrix} 4.9975 \\ 10.0149 \\ 4.9975 \end{bmatrix}$	$\begin{bmatrix} 5.0000 \\ 10.0000 \\ 5.0000 \end{bmatrix}$
$\mu_k = \max\lvert\vec{x}_{k+1}\rvert$	92	6.5870	13.2805	11.3878	10.1394	10.0149	10.0000

In a bit more detail:

$\boldsymbol{\vec{x}_0 = \begin{bmatrix} 5 \\ 2 \\ 9 \end{bmatrix}}$, $\boldsymbol{A\vec{x}_0 = \begin{bmatrix} -12 \\ 92 \\ 8 \end{bmatrix}}$ which we divide by 92 to get…

$\boldsymbol{\vec{x}_1 = \begin{bmatrix} -0.1304 \\ 1.0000 \\ 0.0870 \end{bmatrix}}$, $\boldsymbol{A\vec{x}_1 = \begin{bmatrix} 5.5000 \\ 3.7391 \\ 6.5870 \end{bmatrix}}$ which we divide by 6.5870 to get…

$\boldsymbol{\vec{x}_2 = \begin{bmatrix} 0.8350 \\ 0.5677 \\ 1.0000 \end{bmatrix}}$, $\boldsymbol{A\vec{x}_2 = \begin{bmatrix} 1.1584 \\ 13.2805 \\ 1.9835 \end{bmatrix}}$ which we divide by 13.2805… and so on…

The above is called the Power Method for estimating the dominant (i.e. largest in size, whether positive or negative) eigenvalue and corresponding eigenvector. See page 321.

Let's try the Power Method on another (sort of random) matrix, say $A = \begin{bmatrix} 14 & -1 & -6 \\ 2 & 5 & 0 \\ 10 & -1 & -2 \end{bmatrix}$.

k	0	10	20	30	40
$\vec{x}_k$	$\begin{bmatrix} \\ \\ \end{bmatrix}$	$\begin{bmatrix} \\ \\ \end{bmatrix}$	$\begin{bmatrix} \\ \\ \end{bmatrix}$	$\begin{bmatrix} \\ \\ \end{bmatrix}$	$\begin{bmatrix} \\ \\ \end{bmatrix}$
$\vec{x}_{k+1} = A\vec{x}_k$	$\begin{bmatrix} \\ \\ \end{bmatrix}$	$\begin{bmatrix} \\ \\ \end{bmatrix}$	$\begin{bmatrix} \\ \\ \end{bmatrix}$	$\begin{bmatrix} \\ \\ \end{bmatrix}$	$\begin{bmatrix} \\ \\ \end{bmatrix}$
$\mu_k = \max \lvert \vec{x}_{k+1} \rvert$					

So we estimate the largest eigenvalue of A is _____ with corresponding eigenvector $\vec{v} = \begin{bmatrix} \\ \\ \end{bmatrix}$.

If we do the same process using A^{-1}, then we will find the largest eigenvalue (and its eigenvector) of A^{-1}, which is the *smallest* eigenvalue of A, since $A\vec{v} = \lambda\vec{v} \Rightarrow A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$.

k	0	10	20	30	40
$\vec{x}_k$	$\begin{bmatrix} \\ \\ \end{bmatrix}$	$\begin{bmatrix} \\ \\ \end{bmatrix}$	$\begin{bmatrix} \\ \\ \end{bmatrix}$	$\begin{bmatrix} \\ \\ \end{bmatrix}$	$\begin{bmatrix} \\ \\ \end{bmatrix}$
$\vec{x}_{k+1} = A^{-1}\vec{x}_k$	$\begin{bmatrix} \\ \\ \end{bmatrix}$	$\begin{bmatrix} \\ \\ \end{bmatrix}$	$\begin{bmatrix} \\ \\ \end{bmatrix}$	$\begin{bmatrix} \\ \\ \end{bmatrix}$	$\begin{bmatrix} \\ \\ \end{bmatrix}$
$\mu_k = \max \lvert \vec{x}_{k+1} \rvert$					

On page 323 the book gives an algorithm (optional—we'll not cover it in this class) for finding the eigenvalue closest to any value α, where you can choose what α is.

The above is actually how eigenvalues are computed for larger than 3 x 3 matrices, since determinants for large matrices are horrendously difficult (i.e. time consuming) to compute.

By the way, using the online eigenvalue/eigenvector finder,
the three eigenvalues and eigenvectors of A above are: