Most important ideas:

- Solutions to continuous dynamical system initial value problems.
- Change of variables using eigenvectors and eigenvalues to decouple a system of linear equations.
- What e^A is for a given matrix A and why we care.
- Complex eigenvalues and eigenvectors and their use in solutions to differential equations.
- A comparison of discrete and continuous dynamical systems.

First, an example of **change of variables**, in order to make a more difficult problem into a simpler, solveable problem.

Recall $\int \cos x \; dx = \sin x$ So how to find $\int \sqrt{\sin x} \; \cos x \; dx$?

Let $u = \sin x$, then $du = \cos x \; dx$, and

$$\int \sqrt{\sin x} \; \cos x \; dx = \int \sqrt{u} \, du = \int u^{1/2} \, du = \frac{u^{3/2}}{3/2} = \frac{2}{3} (\sin x)^{3/2}$$

Now a different topic: What does it mean for one quantity to be a function of another quantity?

Sometimes the relationship between two quantities (that is, how one quantity depends on another) is described by (1) how one quantity changes relative to the other, along with (2) some "initial condition" as a starting value.

For example, suppose you know that the population in Malibu is increasing at a rate of 3% per year and is currently (i.e. in the year 2018) 10,000. That is,

$$\frac{dp}{dt} = .03p \,, \quad p(0) = 10{,}000$$

where $p(t)$ is the population t years after 2018. Can we find a formula (a function) which tells us what the population is in any year (for any time t)?

Rewrite $\frac{dp}{dt} = .03p$ as $\frac{dp}{p} = .03dt$, integrate $\int \frac{dp}{p} = \int .03dt$ to get $ln\, p = .03t + C$, and then solve for population p as a function of time t :

$$p = e^{.03t+C} = e^C e^{.03t} \text{ and since } C \text{ is just some unknown constant, so is } e^C.$$

So we have $p(t) = Ce^{.03t}$. You can check that $\frac{dp}{dt} = .03p$.

Finally, use the initial condition $p(0) = 10000$ to find the constant $p(0) = Ce^{.03(0)} = C = 10000$.

So we have found that the solution to the initial value problem

$$\frac{dp}{dt} = .03p \,, \;\; p(0) = 10{,}000$$

is

$$p(t) = 10{,}000e^{.03t} \,.$$

In general the solution to the initial value problem $\frac{dp}{dt} = \lambda p,\ p(0) = p_0$ **is** $p(t) = p_0 e^{\lambda t}$ **.**

Of course it doesn't really matter what we call the variables (the letters).

$$\frac{dy}{dt} = \lambda y\,, \quad y(0) = y_0 \quad \Rightarrow \quad y(t) = y_0 e^{\lambda t}$$

or

$$\frac{dx}{dt} = \lambda x\,, \quad x(0) = x_0 \quad \Rightarrow \quad x(t) = x_0 e^{\lambda t}$$

We will use x or p to represent the (unknown) function of time t in Section 5.7.

Suppose we have more than one population. Say we have two populations p_1 and p_2, and that:

The solution to $\frac{dp_1}{dt} = .03\, p_1\,, \quad p_1(0) = 10{,}000$ **is** $p_1(t) = 10{,}000e^{.03t}$

The solution to $\frac{dp_2}{dt} = .05\, p_2\,, \quad p_2(0) = 75{,}000$ **is** $p_2(t) = 75{,}000e^{.05t}$

We can write this as a single equation:

$$\begin{bmatrix} \frac{dp_1}{dt} \\ \frac{dp_2}{dt} \end{bmatrix} = \begin{bmatrix} .03\, p_1 \\ .05\, p_2 \end{bmatrix} = \begin{bmatrix} .03 & 0 \\ 0 & .05 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \quad \begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix} = \begin{bmatrix} 10{,}000 \\ 75{,}000 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} = \begin{bmatrix} 10{,}000e^{.03t} \\ 75{,}000e^{.05t} \end{bmatrix}$$

That is, in general, for two populations $\vec{p}(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$:

The solution to $\frac{d\vec{p}}{dt} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \vec{p}\,, \quad \vec{p}(0) = \begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ **is** $\vec{p}(t) = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}$.

Or using $\vec{y}$ as the function and d_1, d_2 as the initial (i.e. $t = 0$) values of $y_1(t), y_2(t)$:

The solution to $\frac{d\vec{y}}{dt} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \vec{y}\,, \quad \vec{y}(0) = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ **is** $\vec{y}(t) = \begin{bmatrix} d_1 e^{\lambda_1 t} \\ d_2 e^{\lambda_2 t} \end{bmatrix}$.

It's easy to see how this would generalize to larger than 2×2 linear systems of differential equations.

So what the heck do you do if you are trying to solve the problem (find $\vec{x}(t)$) for

$$\frac{d\vec{x}}{dt} = A\vec{x}\,, \quad \vec{x}_0 = \vec{x}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

where A is not diagonal? How do we change this more difficult problem into a simpler one?

With a change of variables. What sort of change?

Thank goodness once again for **eigenvectors and eigenvalues.**

Suppose $A = PDP^{-1}$ where the columns of P are the (linearly independent) eigenvectors of A and D contains the eigenvalues of A.

Then we have

$$\boldsymbol{\frac{d\vec{x}}{dt} = A\vec{x} = PDP^{-1}\,\vec{x}\,,\quad \vec{x}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}.$$

Multiplying everything on the left by P^{-1} results in

$$P^{-1}\,\frac{d\vec{x}}{dt} = DP^{-1}\vec{x}\,,\quad P^{-1}\,\vec{x}(0) = P^{-1}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Here's our change of variable: Let $\boldsymbol{\vec{y} = P^{-1}\vec{x}}$ (so $\boldsymbol{P\,\vec{y} = \vec{x}}$)

Then $\frac{d\vec{y}}{dt} = \frac{d}{dt}(P^{-1}\,\vec{x}) = P^{-1}\,\frac{d\vec{x}}{dt}$ (since P^{-1} is a matrix of constant values) so we have

$$\frac{d\vec{y}}{dt} = D\vec{y} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\vec{y}\;,\quad \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \boldsymbol{\vec{y}(0) = P^{-1}\,\vec{x}(0)} = P^{-1}\,\vec{x}_0 = P^{-1}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

It's easy to find the solution for this, as we saw on page 2:

$$\boldsymbol{\frac{d\vec{y}}{dt} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\vec{y}\,,\quad \vec{y}(0) = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \quad\Rightarrow\quad \vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} d_1\,e^{\lambda_1 t} \\ d_2\,e^{\lambda_2 t} \end{bmatrix}}.$$

Where $P = [\,\vec{v}_1\;\;\vec{v}_2\,]$, since $\vec{y} = P^{-1}\vec{x}$, we have (changing variables back from $\vec{y}$ to $\vec{x}$)

$$\boldsymbol{\vec{x}(t) = P\,\vec{y}(t) = [\,\vec{v}_1\;\;\vec{v}_2\,]\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = [\,\vec{v}_1\;\;\vec{v}_2\,]\begin{bmatrix} d_1\,e^{\lambda_1 t} \\ d_2\,e^{\lambda_2 t} \end{bmatrix} = d_1\,e^{\lambda_1 t}\,\vec{v}_1 + d_2\,e^{\lambda_2 t}\,\vec{v}_2}.$$

The above is described as "decoupling" (unjoining of) the system of equations.

Example: $\begin{aligned} \frac{dx_1}{dt} &= -x_1 + 10x_2 \\ \frac{dx_2}{dt} &= \;\;5x_1 + \;\;4x_2 \end{aligned}$ i.e. $\frac{d\vec{x}}{dt} = Ax$ where $A = \begin{bmatrix} -1 & 10 \\ 5 & 4 \end{bmatrix}$.

The eigenvectors/eigenvalues of A are $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda_1 = -6,\ \lambda_2 = 9$.

So $$\boldsymbol{\vec{x}(t) = d_1\,e^{\lambda_1 t}\,\vec{v}_1 + d_2\,e^{\lambda_2 t}\,\vec{v}_2 = d_1\,e^{-6t}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + d_2\,e^{9t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}.$$

Now to find d_1 and d_2. Suppose that $\vec{x}(0) = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$, then (as we already saw above) we have

$$\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \vec{x}(0) = d_1\,e^{-6(0)}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + d_2\,e^{9(0)}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \;\Rightarrow\; \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

So $$\boldsymbol{\vec{x}(t) = -1e^{-6t}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 7e^{9t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2e^{-6t} + 7e^{9t} \\ e^{-6t} + 7e^{9t} \end{bmatrix}}$$

That is, $\boldsymbol{x_1(t) = -2e^{-6t} + 7e^{9t}}$ and $\boldsymbol{x_2(t) = e^{-6t} + 7e^{9t}}$.

Check: $$\boldsymbol{\frac{dx_1}{dt} = 12e^{-6t} + 63e^{9t} = -(-2e^{-6t} + 7e^{9t}) + 10(e^{-6t} + 7e^{9t})} \;✓$$
$$\boldsymbol{\frac{dx_2}{dt} = -6e^{-6t} + 63e^{9t} = 5(-2e^{-6t} + 7e^{9t}) + 4(e^{-6t} + 7e^{9t})} \;✓$$

By the way, now is a good time to (re-)read Section 5.6, p. 306, about change of variable. In 5.6 this change of variable idea was not really needed in order to solve the discrete time problems, but in 5.7 with continuous time problems it is necessary.

Another (really cool) view of all of this…

First, recall the Taylor Series $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$

It turns out that the same is true for a matrix:

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

This would be tough to compute (especially the infinite $\cdots$ part) unless we have a way to easily compute A^k for any k. As usual, it's eigenvectors and eigenvalues that makes this possible.

Recall: if $A = PDP^{-1}$, then $A^k = PD^kP^{-1} = P\begin{bmatrix}\lambda_1^k & 0 \\ 0 & \lambda_2^k\end{bmatrix}P^{-1}$, and we have

$$\begin{aligned}
e^{At} &= I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots \\
&= P\,P^{-1} + P\,Dt\,P^{-1} + \frac{1}{2!}P\,(Dt)^2\,P^{-1} + \frac{1}{3!}P\,(Dt)^3\,P^{-1} + \cdots \\
&= P\left[\,I + (Dt) + \frac{1}{2!}(Dt)^2 + \frac{1}{3!}(Dt)^3 + \cdots\,\right]P^{-1} \\
&= P\left[\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} + \begin{bmatrix}\lambda_1 t & 0 \\ 0 & \lambda_2 t\end{bmatrix} + \frac{1}{2!}\begin{bmatrix}(\lambda_1 t)^2 & 0 \\ 0 & (\lambda_2 t)^2\end{bmatrix} + \frac{1}{3!}\begin{bmatrix}(\lambda_1 t)^3 & 0 \\ 0 & (\lambda_2 t)^3\end{bmatrix} + \cdots\right]P^{-1} \\
&= P\begin{bmatrix}1 + \lambda_1 t + \frac{1}{2!}(\lambda_1 t)^2 + \frac{1}{3!}(\lambda_1 t)^3 + \cdots & 0 \\ 0 & 1 + \lambda_2 t + \frac{1}{2!}(\lambda_2 t)^2 + \frac{1}{3!}(\lambda_3 t)^3 + \cdots\end{bmatrix}P^{-1} \\
&= P\begin{bmatrix}e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t}\end{bmatrix}P^{-1}
\end{aligned}$$

And similarly for larger than 2×2 matrices.

Recall: the solution to the differential equation (also known as an initial value problem)

$$\frac{dx}{dt} = ax, \quad x(0) = x_0$$

Is $x(t) = x_0 e^{at}$.

Then the solution to the multivariable differential equation

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \vec{x}(0) = \vec{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

is

$\vec{x}(t) = e^{At}\vec{x}_0$ (a matrix times a vector)

$$= P \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} P^{-1} \vec{x}_0$$

$$= [\vec{v}_1 \ \vec{v}_2] \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$= [e^{\lambda_1 t}\vec{v}_1 \ \ e^{\lambda_2 t}\vec{v}_2] \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$= d_1 e^{\lambda_1 t}\vec{v}_1 + d_2 e^{\lambda_2 t}\vec{v}_2$$

where $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = P^{-1}\vec{x}_0$.

(Does the above formula for $\vec{x}(t)$ look familiar?)

In our next example we use the fact $\boldsymbol{e^{ib} = \cos b + i \sin b}$ so $\boldsymbol{e^{a+ib} = e^a e^{ib} = e^a(\cos b + i \sin b)}$.

Example 1: Predator/prey example (with complex eigenvalues/vectors): recall $O = Owls, R = Rats$.

$$\begin{aligned} \frac{dO}{dt} &= .1\,O + .2\,R \\ \frac{dR}{dt} &= -.2\,O + .1\,R \end{aligned} \qquad \text{i.e.} \quad \frac{d\vec{p}}{dt} = A\vec{p} \quad \text{where } \vec{p} = \begin{bmatrix} O \\ R \end{bmatrix}, \ A = \begin{bmatrix} .1 & .2 \\ -.2 & .1 \end{bmatrix}.$$

What do the values of A tell us?

The eigenvectors and eigenvalues of A are $\begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $.1 + .2i,\ .1 - .2i$:

$$\begin{bmatrix} .1 & .2 \\ -.2 & .1 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} .1 + .2i \\ -.2 + .1i \end{bmatrix} = (.1 + .2i)\begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} .1 & .2 \\ -.2 & .1 \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} .1 - .2i \\ -.2 - .1i \end{bmatrix} = (.1 - .2i)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Then population $\boldsymbol{\vec{p}(t) = d_1 e^{\lambda_1 t}\vec{v}_1 + d_2 e^{\lambda_2 t}\vec{v}_2 = d_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(.1+.2i)t} + d_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(.1-.2i)t}}$, that is

$$\begin{aligned} O(t) &= d_1(1)e^{(.1+.2i)t} + d_2(1)e^{(.1-.2i)t} \\ &= e^{.1t}[d_1(\cos .2t + i \ \sin .2t) + d_2(\cos .2t - i \ \sin .2t)] \\ &= e^{.1t}[(d_1 + d_2)\cos .2t + (d_1 - d_2)(i)\sin .2t] \end{aligned}$$

$$\begin{aligned} R(t) &= d_1(i)e^{(.1+.2i)t} + d_2(-i)e^{(.1-.2i)t} \\ &= e^{.1t}[d_1(i)(\cos .2t + i \ \sin .2t) + d_2(-i)(\cos .2t - i \ \sin .2t)] \\ &= e^{.1t}[(d_1 - d_2)(i)\cos .2t - (d_1 + d_2)\sin .2t] \end{aligned}$$

So

$$\boldsymbol{\begin{aligned} O(t) &= e^{.1t}[b_1 \cos .2t + b_2 \sin .2t] \\ R(t) &= e^{.1t}[b_2 \cos .2t - b_1 \sin .2t] \end{aligned}}$$

where

$$\begin{aligned} b_1 &= d_1 + d_2 \\ b_2 &= i(d_1 - d_2) \end{aligned}$$

Suppose we have initial conditions of: $O(0) = 20,\ R(0) = 10$

Then:

$$\begin{aligned} 20 = O(0) &= e^0[b_1 \cos 0 + b_2 \sin 0] = b_1 \\ 10 = R(0) &= e^0[b_2 \cos 0 - b_1 \sin 0] = b_2 \end{aligned}$$

So:

$$\boldsymbol{\begin{aligned} O(t) &= e^{.1t}[20 \cos .2t + 10 \sin .2t] \\ R(t) &= e^{.1t}[10 \cos .2t - 20 \sin .2t] \end{aligned}}$$

A bit more generally (but still for this specific type of matrix A), where

$$\frac{dO}{dt} = a\,O + b\,R \qquad \text{i.e. } \frac{d\vec{p}}{dt} = A\vec{p} \;\text{ where } \vec{p} = \begin{bmatrix} O \\ R \end{bmatrix},\; A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$\frac{dR}{dt} = -b\,O + a\,R$$

and $O(0) = O_0$, $R(0) = R_0$, the solution is

$$O(t) = e^{at}[O_0 \cos\, bt + R_0 \sin\, bt]$$

$$R(t) = e^{at}[R_0 \cos\, bt - O_0 \sin\, bt]$$

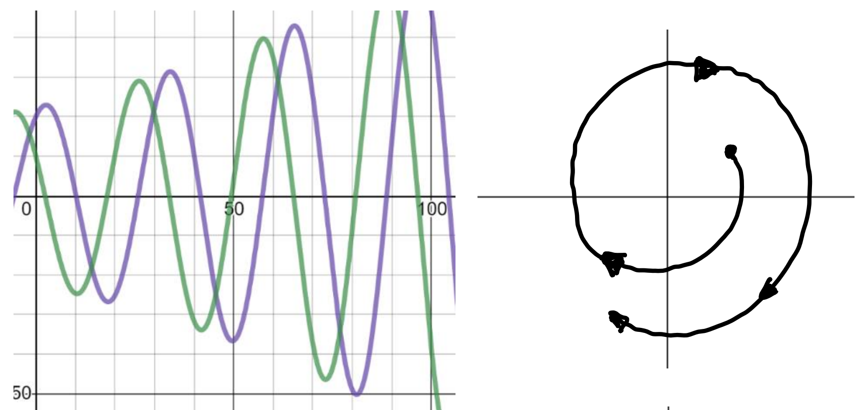
A few interesting cases, with $O_0 = 20$, $R_0 = 10$.

$a > 0, b > 0$, e.g. $a = .01$, $b = .2$

$O(t) = e^{.01t}[20 \cos\, .2t + 10 \sin\, .2t]$

$R(t) = e^{.01t}[10 \cos\, .2t - 20 \sin\, .2t]$

(And what does $a = .01$ mean?)

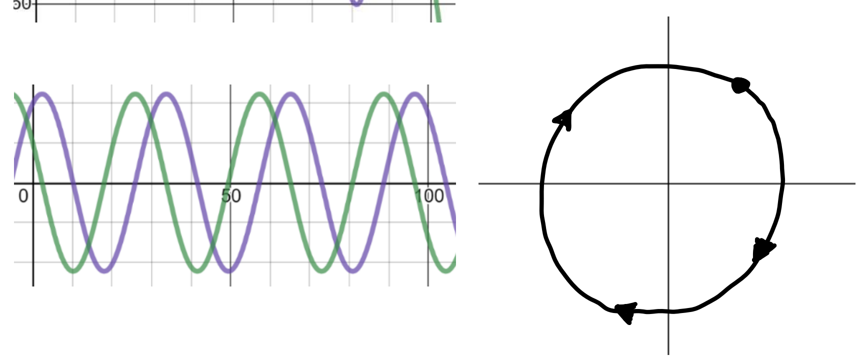


$a = 0, b > 0$, e.g. $a = 0$, $b = .2$

$O(t) = 20 \cos\, .2t + 10 \sin\, .2t$

$R(t) = 10 \cos\, .2t - 20 \sin\, .2t$

(And what does $a = 0$ mean?)

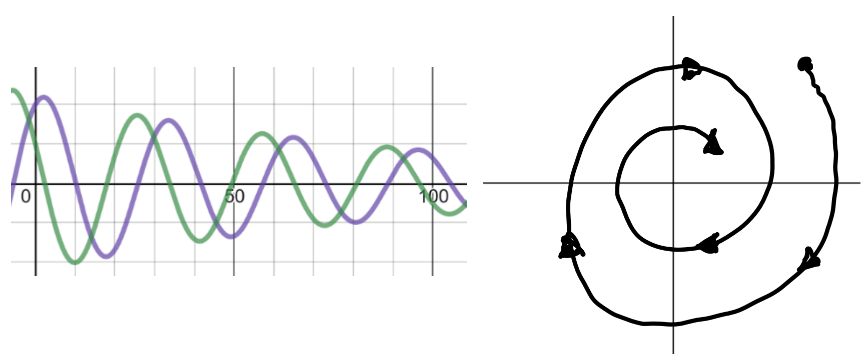


$a < 0, b > 0$, e.g. $a = -.01$, $b = .2$

$O(t) = e^{-.01t}[20 \cos\, .2t + 10 \sin\, .2t]$

$R(t) = e^{-.01t}[10 \cos\, .2t - 20 \sin\, .2t]$

(And what does $a = -.01$ mean?)



If $b < 0$, then the predator and the prey have simply swapped roles: the hunter becomes the hunted and conversely. If $b = 0$ then there is no interaction between owls and rats.

Finally, a few thoughts before we compare the discrete and continuous cases.

First, if $A\vec{v} = \lambda\vec{v}$, then $(A - cI)\vec{v} = A\vec{v} - c\vec{v} = (\lambda - c)\vec{v}$. That is, if λ is an eigenvalue of A, then $\lambda - c$ is an eigenvalue of $A - cI$ and with the same eigenvector $\vec{v}$.

Example:

$$A \quad \vec{v} = \quad \lambda \ \vec{v} \qquad\qquad (A - I)\,\vec{v} = (\lambda - 1)\vec{v}$$

$$\begin{bmatrix} 1 & 6 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad\qquad \begin{bmatrix} 0 & 6 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = -2\begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad\qquad \begin{bmatrix} 0 & 6 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = -3\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Second, a comparison of the discrete and continuous cases for *one* population which grows at 2% per year with initial population of p_0.

Discrete: $\boldsymbol{p_{k+1} = 1.02\, p_k \Rightarrow\ p_k = (1.02)^k\, p_0 = p_0\,(1.02)^k}$

Continuous: $\boldsymbol{\frac{dp}{dt} = .02\,p \quad \Rightarrow p(t) = e^{.02t}\, p_0}$ **(or $\boldsymbol{p_0\, e^{.02t}}$, if you prefer).**

Third, remember the Taylor Series $\boldsymbol{e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \ \approx\ 1 + t}$ if t is small.

So, for example, $e^{.02} \approx 1.02$, so $(e^{.02})^t \approx (1.02)^t$, i.e. $\boldsymbol{e^{.02t} \approx (1.02)^t}$.

Note: $e^{.02t}$ is actually slightly $> (1.02)^t$, especially as t gets larger. This is analogous to how if you earn 2% interest per year in the bank, 2% *continuously compounded* interest will result in more interest earned than 2% earned *once a year*. Indeed, $e^{.02t} = (e^{.02})^t = 1.0202^t$. So 2% continuous growth is like 2.02% growth that is experienced once a year.

So $e^{.02t} \approx (1 + .02)^t = (1.02)^t$, and similarly, $e^{-.42t} \approx (1 - .42)^t = 0.58^t$. We'll see both of these in the following example.

Discrete Example 2 (Example 1 from Book Section 5.6, as well as Example 1 from Handout 5.6):

Where $\begin{bmatrix} O \\ R \end{bmatrix}_0 = \begin{bmatrix} 35 \\ 29 \end{bmatrix}$ and $\begin{matrix} O_{k+1} = & .5O_k + \ .4R_k \\ R_{k+1} = & -.104O_k + 1.1R_k \end{matrix}$, that is, $\begin{bmatrix} O \\ R \end{bmatrix}_{k+1} = \begin{bmatrix} .5 & .4 \\ -.104 & 1.1 \end{bmatrix} \begin{bmatrix} O \\ R \end{bmatrix}_k$.

Then $\begin{bmatrix} O \\ R \end{bmatrix}_k = c_1(1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} + c_1(.58)^k \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ where $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 13 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 35 \\ 29 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

That is, $\begin{bmatrix} O \\ R \end{bmatrix}_k = 2(1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} + 3(.58)^k \begin{bmatrix} 5 \\ 1 \end{bmatrix} = (1.02)^k \begin{bmatrix} 20 \\ 26 \end{bmatrix} + (.58)^k \begin{bmatrix} 15 \\ 3 \end{bmatrix}$.

where k is the (discrete) number of months that have gone by.

The continuous analog of that example:

Where $\begin{bmatrix} O \\ R \end{bmatrix}_0 = \begin{bmatrix} 35 \\ 29 \end{bmatrix}$ and $\begin{matrix} \frac{dO}{dt} = & -.5\,O + \ .4\,R \\ \frac{dR}{dt} = & -.104\,O + 0.1\,R \end{matrix}$, that is, $\frac{d\vec{x}}{dt} = \begin{bmatrix} -.5 & .4 \\ -.104 & 0.1 \end{bmatrix} \vec{x}$, where $\vec{x} = \begin{bmatrix} O \\ R \end{bmatrix}$.

Then $\vec{x}(t) = \begin{bmatrix} O(t) \\ R(t) \end{bmatrix} = d_1\, e^{.02t} \begin{bmatrix} 10 \\ 13 \end{bmatrix} + d_2 e^{-.42t} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ where $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 13 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 35 \\ 29 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

That is, $\vec{x}(t) = \begin{bmatrix} O(t) \\ R(t) \end{bmatrix} = 2e^{.02t} \begin{bmatrix} 10 \\ 13 \end{bmatrix} + 3e^{-.42t} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = e^{.02t} \begin{bmatrix} 20 \\ 26 \end{bmatrix} + e^{-.42t} \begin{bmatrix} 15 \\ 3 \end{bmatrix}$.

where t is (continuous) amount of time in months that has gone by.