Most important ideas:

- Solutions to <u>discrete</u> (continuous in 5.7) dynamical system initial value problems.
- Complex eigenvalues and eigenvectors and their use in solutions to differential equations.

Let's spend a few minutes discussing what in R^2

$$A^k\vec{v} = A^k(c_1\vec{v}_1 + c_2\vec{v}_2) = \lambda_1^k c_1\vec{v}_1 + \lambda_2^k c_2\vec{v}_2$$

looks like, where $\vec{v}_1, \vec{v}_2$ are eigenvectors of A.

Let's look at Book Example 1 to get familiar with the sort of problem we'll see in this section. For now, we'll not look at the details of how they deal with this problem. For now, let's just see how the populations change. For now, let's use their given value of $p = .104$. So

$\begin{bmatrix} O_{k+1} \\ R_{k+1} \end{bmatrix} = \begin{bmatrix} .5 & .4 \\ -.104 & 1.1 \end{bmatrix}\begin{bmatrix} O_k \\ R_k \end{bmatrix}$, that is, $\vec{x}_{k+1} = A\vec{x}_k$ where $\vec{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$ and $A = \begin{bmatrix} .5 & .4 \\ -.104 & 1.1 \end{bmatrix}$.

Let's discuss what the four values in A mean.

Suppose the initial owl and rat populations are $\vec{x}_0 = \begin{bmatrix} 200 \\ 60 \end{bmatrix}$. Then after one month we have

$$\vec{x}_1 = A\vec{x}_0 = \begin{bmatrix} .5 & .4 \\ -.104 & 1.1 \end{bmatrix}\begin{bmatrix} 200 \\ 60 \end{bmatrix} = \begin{bmatrix} 124.0 \\ 45.2 \end{bmatrix}$$

and after two months

$$\vec{x}_2 = A\vec{x}_1 = \begin{bmatrix} .5 & .4 \\ -.104 & 1.1 \end{bmatrix}\begin{bmatrix} 124.0 \\ 45.2 \end{bmatrix} \approx \begin{bmatrix} 80.1 \\ 36.8 \end{bmatrix}$$

and so on to get

k	0	1	2	3	4	5	10	15	20	25
$\vec{x}_k$	$\begin{bmatrix} 200.0 \\ 60.0 \end{bmatrix}$	$\begin{bmatrix} 124.0 \\ 45.2 \end{bmatrix}$	$\begin{bmatrix} 80.1 \\ 36.8 \end{bmatrix}$	$\begin{bmatrix} 54.8 \\ 32.2 \end{bmatrix}$	$\begin{bmatrix} 40.3 \\ 29.7 \end{bmatrix}$	$\begin{bmatrix} 32.0 \\ 28.5 \end{bmatrix}$	$\begin{bmatrix} 22.9 \\ 29.0 \end{bmatrix}$	$\begin{bmatrix} 24.5 \\ 31.8 \end{bmatrix}$	$\begin{bmatrix} 27.0 \\ 35.1 \end{bmatrix}$	$\begin{bmatrix} 29.8 \\ 38.8 \end{bmatrix}$

These points are plotted on the next page. In that plot, the first value in each population vector is like the x value and the second value is like the y value. These are the top right set of points, the purple ones if you have this in color. I also have plotted five other sets of population points, which I'll not list in detail, but here are the six initial populations used:

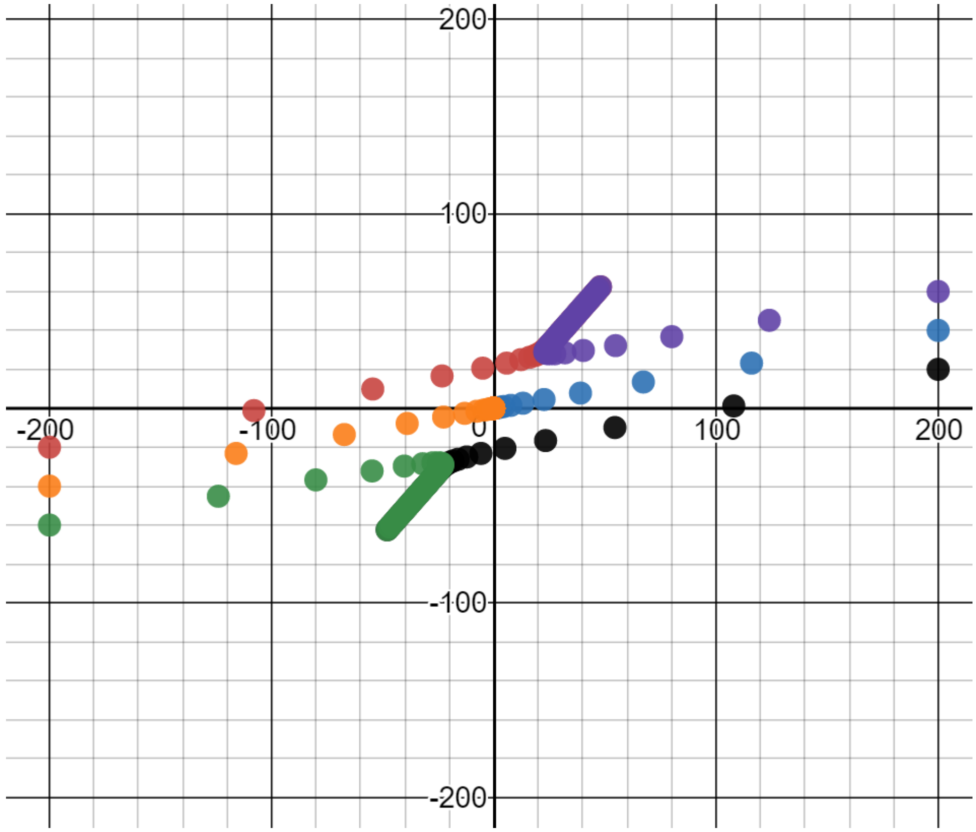
$$\begin{bmatrix} \mathbf{200} \\ \mathbf{60} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{200} \\ \mathbf{40} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{200} \\ \mathbf{20} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{-200} \\ \mathbf{-60} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{-200} \\ \mathbf{-40} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{-200} \\ \mathbf{-20} \end{bmatrix}$$

The last three with negative populations don't make sense, of course, but they help us see the various paths the population points take, depending on the initial populations.

It seems like there are certain paths that these vectors follow. A few questions:

1. What exactly are these paths, and how are they related to A and/or $\vec{x}_0$?
2. Why are the population points moving toward or away from the origin?
3. Why are there different "speeds" at which these points seem to move?

The short answers: eigenvalues and eigenvectors, which we'll see a little later. In the second plot, I've plotted just the two lines/paths that the population points seem to be following: the eigenvectors of A.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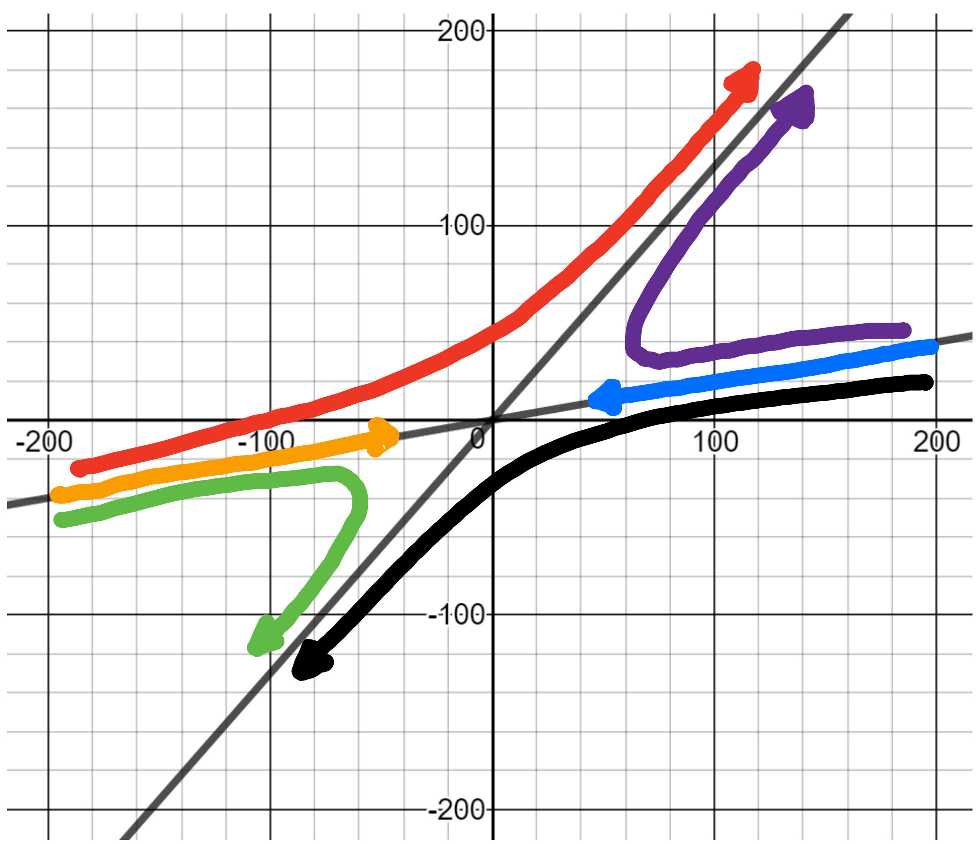
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Recall that if $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are eigenvectors of A corresponding to eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, and if $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$, then

$$A\vec{v} = A(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) = \lambda_1 c_1\vec{v}_1 + \lambda_2 c_2\vec{v}_2 + \cdots + \lambda_n c_n\vec{v}_n$$

and

$$A^k\vec{v} = A^k(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) = \lambda_1^k c_1\vec{v}_1 + \lambda_2^k c_2\vec{v}_2 + \cdots + \lambda_n^k c_n\vec{v}_n\,.$$

With this in mind, let's think about what is really going on in this example. The eigenvectors and corresponding eigenvalues of $A = \begin{bmatrix} .5 & .4 \\ -.104 & 1.1 \end{bmatrix}$ are $\begin{bmatrix} 10 \\ 13 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $1.02, 0.58$.

Since

$$\vec{x}_0 = \begin{bmatrix} 200 \\ 60 \end{bmatrix} = \tfrac{20}{11}\begin{bmatrix} 10 \\ 13 \end{bmatrix} + \tfrac{400}{11}\begin{bmatrix} 5 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 18.20 \\ 23.64 \end{bmatrix} + \begin{bmatrix} 181.82 \\ 36.36 \end{bmatrix}$$

then

$$\vec{x}_k \approx (1.02)^k \begin{bmatrix} 18.20 \\ 23.64 \end{bmatrix} + (0.58)^k \begin{bmatrix} 181.82 \\ 36.36 \end{bmatrix}$$

For example,

$$\vec{x}_{10} \approx (1.02)^{10} \begin{bmatrix} 18.20 \\ 23.64 \end{bmatrix} + (0.58)^{10} \begin{bmatrix} 181.82 \\ 36.36 \end{bmatrix} \approx \begin{bmatrix} 22.9 \\ 29.0 \end{bmatrix}$$

The fact that

$$\vec{x}_k = (1.02)^k \begin{bmatrix} 18.20 \\ 23.64 \end{bmatrix} + (0.58)^k \begin{bmatrix} 181.82 \\ 36.36 \end{bmatrix}$$

is somewhat useful in allowing us to more quickly (with less work) compute $\vec{x}_k$, but its real importance is that it helps us understand what paths the population points take, depending on the initial populations $\vec{x}_0$. Let's talk about this for Book Example 1 in the plots on the previous page. The two paths (the two lines in the second plot) are simply the eigenvectors of A.

Let's look at Book Example 2 (with Figure 1), Example 3 (with Figure 2), and Example 4 (with Figure 3). The eigenvalues of diagonal matrices are simply the diagonal values, and the eigenvectors are simply the standard bases vectors (a.k.a. the columns of the identity matrix). So in this case, the paths along which the population points move are simply the x and y axes, since $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the eigenvectors of the 2×2 diagonal matrices in these examples.

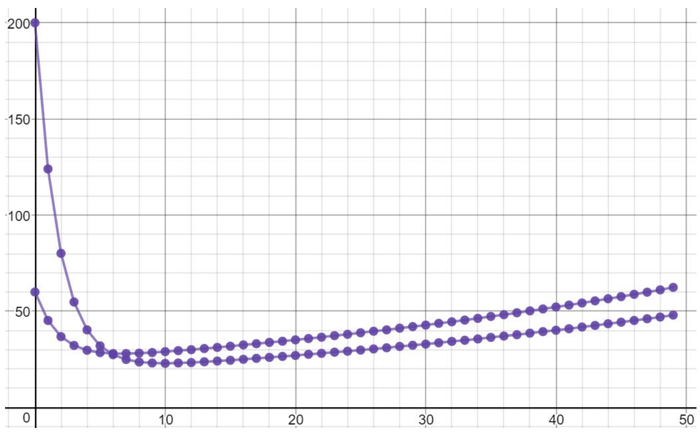
In these problems we see answers to the second and third questions about why the paths move toward or away from the origin and why the rate of change is faster or slower along those paths:

- The spacing of the points tells us how quickly the points are moving, how quickly the population amounts are changing. Larger gaps mean more rapid movement, which we see in the examples results from more extreme eigenvalues:
  - If the value is > 1, then the points move away from the origin, and the larger the value, the faster the movement of the points away from the origin.
  - If the value is $0 < \quad < 1$, then the points move toward the origin, and the smaller the value (the closer to 0), the faster the movement of the points toward the origin.

Let's think about what these values mean in the diagonal matrices in Book Examples 2 – 4. So this is like we saw in Book Example 1, except that the paths the points followed in Example 1 are not simply the x and y axes, they are the eigenvectors of A and the eigenvalues determine whether the movement along each "eigenvector path/axis" is toward or away from the origin.

Here is another way we could have plotted the two populations, each as a function of month. Here are the vectors we had found.

k	0	1	2	3	4	5	10	15	20	25
$\vec{x}_k$	$\begin{bmatrix}200.0\\60.0\end{bmatrix}$	$\begin{bmatrix}124.0\\45.2\end{bmatrix}$	$\begin{bmatrix}80.1\\36.8\end{bmatrix}$	$\begin{bmatrix}54.8\\32.2\end{bmatrix}$	$\begin{bmatrix}40.3\\29.7\end{bmatrix}$	$\begin{bmatrix}32.0\\28.5\end{bmatrix}$	$\begin{bmatrix}22.9\\29.0\end{bmatrix}$	$\begin{bmatrix}24.5\\31.8\end{bmatrix}$	$\begin{bmatrix}27.0\\35.1\end{bmatrix}$	$\begin{bmatrix}29.8\\38.8\end{bmatrix}$



On day 2 we'll talk about complex eigenvalues. This is an interesting case, both mathematically and in terms of what it means in how the owl and rat populations are changing.

First, recall the Taylor Series $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots$$

So

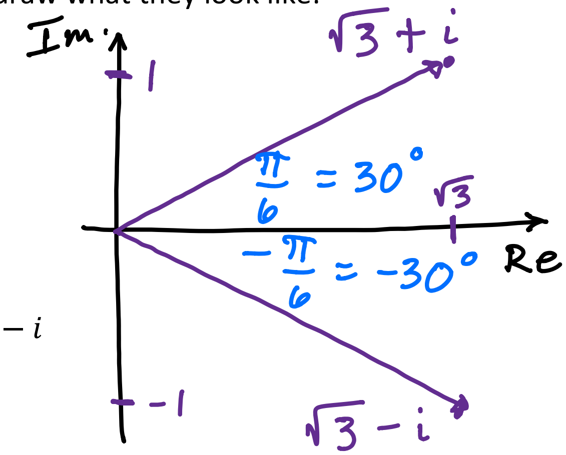
$$e^{ix} = 1 + ix + \frac{1}{2!}(ix)^2 + \frac{1}{3!}(ix)^3 + \frac{1}{4!}(ix)^4 + \frac{1}{5!}(ix)^5 + \cdots$$

$$= \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right) + i\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right) = \cos x + i \sin x.$$

Here are two examples that arise later in our examples—and I'll draw what they look like:

$$2e^{i\left(\frac{\pi}{6}\right)} = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right) = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \sqrt{3} + i$$

$$2e^{i\left(-\frac{\pi}{6}\right)} = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \sqrt{3} - i$$



Another very interesting example is that $e^{i\pi} = \cos\pi + i\sin\pi = -1$, so that $e^{i\pi} + 1 = 0$. A few years back, this was voted the #1 equation of all time in a poll run by the NY Times.

Now on to our predator/prey example with complex eigenvalues/vectors:

$$O_{k+1} = \frac{\sqrt{3}}{2}\, O_k + \frac{1}{2}\, R_k \approx .866\, O_k + .5\, R_k$$
$$R_{k+1} = -\frac{1}{2}\, O_k + \frac{\sqrt{3}}{2}\, R_k \approx -.5\, O_k + .866\, R_k$$

$$\text{i.e. } \vec{x}_{k+1} = A\vec{x}_k \quad \text{where } \vec{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}, \quad A = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

In class, let's discuss the significance of the four values in A.

The eigenvectors and eigenvalues of A are $\begin{bmatrix} 1 \\ i \end{bmatrix}$, $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\frac{\sqrt{3}}{2} + \frac{i}{2}$, $\frac{\sqrt{3}}{2} - \frac{i}{2}$.

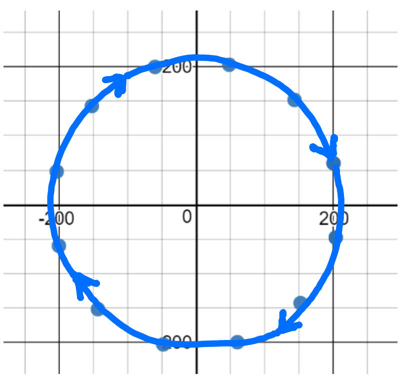
(Notice that the size of both eigenvalues is $\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1$.)

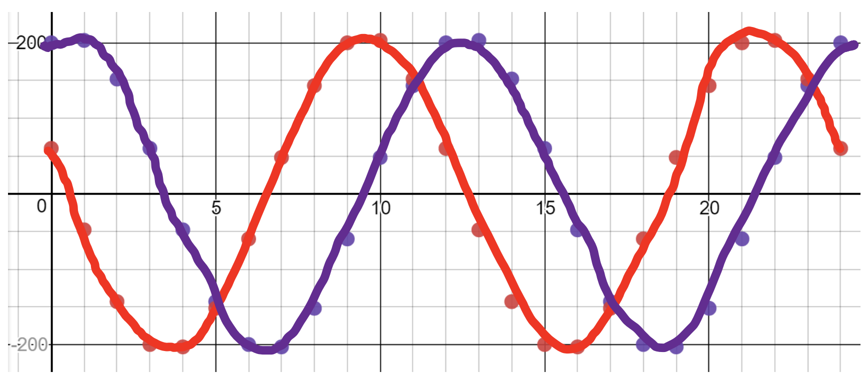
$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}+i}{2} \\ \frac{-1+\sqrt{3}\,i}{2} \end{bmatrix} = \frac{\sqrt{3}+i}{2}\begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}-i}{2} \\ \frac{-1-\sqrt{3}\,i}{2} \end{bmatrix} = \frac{\sqrt{3}-i}{2}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Before we jump into the details of this problem, let's just plot a population path for some initial population. Suppose $\vec{x}_0 = \begin{bmatrix} O_0 \\ R_0 \end{bmatrix} = \begin{bmatrix} 200 \\ 60 \end{bmatrix}$. Then we have (after rounding a bit):

k	0	1	2	3	4	5	6	7	8	9	10	11	12
$\vec{x}_k$	$\begin{bmatrix} 200 \\ 60 \end{bmatrix}$	$\begin{bmatrix} 203 \\ -48 \end{bmatrix}$	$\begin{bmatrix} 152 \\ -143 \end{bmatrix}$	$\begin{bmatrix} 60 \\ -200 \end{bmatrix}$	$\begin{bmatrix} -48 \\ -203 \end{bmatrix}$	$\begin{bmatrix} -143 \\ -152 \end{bmatrix}$	$\begin{bmatrix} -200 \\ -60 \end{bmatrix}$	$\begin{bmatrix} -203 \\ 48 \end{bmatrix}$	$\begin{bmatrix} -152 \\ 143 \end{bmatrix}$	$\begin{bmatrix} -60 \\ 200 \end{bmatrix}$	$\begin{bmatrix} 48 \\ 203 \end{bmatrix}$	$\begin{bmatrix} 143 \\ 152 \end{bmatrix}$	$\begin{bmatrix} 200 \\ 60 \end{bmatrix}$

which we would plot in either of the two ways we've seen earlier.





These kind of look like *sine* or *cosine* functions, don't they? What's that all about? We'll see.

Note that $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ for $\phi = -30°$, so A is the matrix that rotates the vector which it is multiplying by $-30°$. See p. 72 in the book. So after 12 rotations, the resulting population vectors (due to multiplying by A) repeat.

The result of complex (imaginary) eigenvalues oscillation in the populations. This is the natural behavior of populations that are competing, such as predators and prey. Of course, negative populations don't make sense—the equations given above would actually be a little different than what is given—but this oscillatory behavior would still occur.

Let's modify our problem a bit (just multiply all values in A by 2). So we have

$$\begin{aligned} O_{k+1} &= \sqrt{3}\,O_k + \;\,1\,R_k \\ R_{k+1} &= -1\,O_k + \sqrt{3}\,R_k \end{aligned} \qquad \text{i.e. } \vec{x}_{k+1} = A\vec{x}_k \quad \text{where } \vec{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix},\; A = \begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}$$

The eigenvectors and eigenvalues of A are $\begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\sqrt{3}+i,\; \sqrt{3}-i$:

$$\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} \sqrt{3}+i \\ -1+\sqrt{3}\,i \end{bmatrix} = (\sqrt{3}+i)\begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \sqrt{3}-i \\ -1-\sqrt{3}\,i \end{bmatrix} = (\sqrt{3}-i)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Where $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$, then $\vec{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix} = A^k\vec{x}_0 = c_1\left(\sqrt{3}+i\right)^k\begin{bmatrix} 1 \\ i \end{bmatrix} + c_2\left(\sqrt{3}-i\right)^k\begin{bmatrix} 1 \\ -i \end{bmatrix}$, so

$$O_k = c_1(1)\left(\sqrt{3}+i\right)^k + c_2(1)\left(\sqrt{3}-i\right)^k$$

$$= c_1(1)\left(2e^{i\left(\frac{\pi}{6}\right)}\right)^k + c_2(1)\left(2e^{i\left(-\frac{\pi}{6}\right)}\right)^k$$

$$= c_1\left(2^k e^{i\left(k\left(\frac{\pi}{6}\right)\right)}\right) + c_2\left(2^k e^{i\left(-k\left(\frac{\pi}{6}\right)\right)}\right)$$

$$= c_1\left(2^k\left(\cos\left(k\left(\frac{\pi}{6}\right)\right) + i\sin\left(k\left(\frac{\pi}{6}\right)\right)\right)\right) + c_2\left(2^k\left(\cos\left(-k\left(\frac{\pi}{6}\right)\right) + i\sin\left(-k\left(\frac{\pi}{6}\right)\right)\right)\right)$$

$$= c_1\left(2^k\cos\left(k\left(\frac{\pi}{6}\right)\right) + i\,2^k\sin\left(k\left(\frac{\pi}{6}\right)\right)\right) + c_2\left(2^k\cos\left(k\left(\frac{\pi}{6}\right)\right) - i\,2^k\sin\left(k\left(\frac{\pi}{6}\right)\right)\right)$$

$$= (c_1+c_2)\left(2^k\cos\left(k\left(\frac{\pi}{6}\right)\right)\right) + i(c_1-c_2)\left(2^k\sin\left(k\left(\frac{\pi}{6}\right)\right)\right).$$

Recall:

$$\cos(-\theta) = \cos\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$R_k = c_1(i)\left(\sqrt{3}+i\right)^k + c_2(-i)\left(\sqrt{3}-i\right)^k$$

$$= c_1(i)\left(2e^{i\left(\frac{\pi}{6}\right)}\right)^k + c_2(-i)\left(2e^{i\left(-\left(\frac{\pi}{6}\right)\right)}\right)^k$$

$$= c_1\left(i\,2^k\,e^{i\left(k\left(\frac{\pi}{6}\right)\right)}\right) - c_2\left(i\,2^k e^{i\left(-k\left(\frac{\pi}{6}\right)\right)}\right)$$

$$= c_1\left(i\,2^k\left(\cos\left(k\left(\frac{\pi}{6}\right)\right) + i\sin\left(k\left(\frac{\pi}{6}\right)\right)\right)\right) - c_2\left(i\,2^k\left(\cos\left(-k\left(\frac{\pi}{6}\right)\right) + i\sin\left(-k\left(\frac{\pi}{6}\right)\right)\right)\right)$$

$$= c_1\left(i\,2^k\cos\left(k\left(\frac{\pi}{6}\right)\right) - 2^k\sin\left(k\left(\frac{\pi}{6}\right)\right)\right) - c_2\left(i\,2^k\cos\left(k\left(\frac{\pi}{6}\right)\right) + 2^k\sin\left(k\left(\frac{\pi}{6}\right)\right)\right)$$

$$= i(c_1-c_2)\left(2^k\cos\left(k\left(\frac{\pi}{6}\right)\right)\right) - (c_1+c_2)\left(2^k\sin\left(k\left(\frac{\pi}{6}\right)\right)\right).$$

So

$$O_k = 2^k[b_1\cos(k\left(\frac{\pi}{6}\right)) + b_2\sin(k\left(\frac{\pi}{6}\right))]$$

$$R_k = 2^k[b_2\cos(k\left(\frac{\pi}{6}\right)) - b_1\sin(k\left(\frac{\pi}{6}\right))]$$

where

$$b_1 = \quad c_1 + c_2$$

$$b_2 = i(c_1 - c_2)$$

Note that for $k = 0$ we have

$$O_0 = 2^0\left[b_1\cos\left(0\left(\frac{\pi}{6}\right)\right) + b_2\sin\left(0\left(\frac{\pi}{6}\right)\right)\right] = b_1$$

$$R_0 = 2^0\left[b_2\cos\left(0\left(\frac{\pi}{6}\right)\right) - b_1\sin\left(0\left(\frac{\pi}{6}\right)\right)\right] = b_2$$

So this tells us how to find b_1 and b_2 in the formulas for O_k and R_k above: they are simply the initial populations. So after wading through all of the above messy details, this model turns out to be pretty simple. Again, the periodic (oscillating) nature of the populations is because the eigenvalues are complex.

A real (all of its values/entries are real numbers) matrix of any size with complex eigenvalues will have those eigenvalues in complex conjugate pairs, e.g. $\lambda_1 = \sqrt{3} + i \Rightarrow \lambda_2 = \overline{\lambda_1} = \sqrt{3} - i$ as above or $0.9 \pm .2i$, as in Book Example 6. This is also true of the eigenvectors, e.g. $\vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \Rightarrow \vec{v}_2 = \overline{\vec{v}_1} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$, as above.

Complex eigenvalues mean rotation, as seen above. In the simple case of 2×2 matrices, the eigenvalues determine the rotation angle. Also, the size of the eigenvalues determines whether the populations are increasing (if $|\lambda| > 1$) or decreasing (if $|\lambda| < 1$) or not changing (if $|\lambda| = 1$) as the rotations occur. All of these statements are exactly true if A is a rotation matrix (or some multiple of one), as on page 72, and approximately true otherwise, as in Book Example 6.

Let's look at the two examples above, and then Book Example 6.

This is more complicated if there are more populations, but complex eigenvalues still are what lead to oscillation in the populations. In the book they briefly discuss this beginning at the bottom of page 307, but in our discussion we're not going to get into things beyond the 2×2 case.

The discussion regarding the change of variable P on page 306 is more important and more pertinent to Section 5.7, so I'll wait until then for us to discuss it in class. There is already enough going on in this section without. In this section, it's of interest. In the next section, we simply cannot do what we need to do without it.