Most important ideas:

- Diagonalization of a matrix. This is very useful and helps us really understand what a particular matrix "is" and what is happening when you multiply a vector by that matrix.

Example 1: $D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$, $\boldsymbol{D^2 = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}}$. Similarly, $\boldsymbol{D^k = \begin{bmatrix} 2^k & 0 \\ 0 & (-3)^k \end{bmatrix}}$.

In general, for diagonal matrix $D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$, $\boldsymbol{D^k = \begin{bmatrix} d_1^k & & & \\ & d_2^k & & \\ & & \ddots & \\ & & & d_n^k \end{bmatrix}}$.

We like diagonal matrices for a plethora of reasons, including this one.

Next, suppose that $n \times n$ matrix A has n distinct (i.e. linearly independent) eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ with corresponding eigenvalues (not necessarily all different) $\lambda_1, \lambda_2, \dots, \lambda_n$. Then:

$$\boldsymbol{A[\,\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n\,]} = [\,A\vec{v}_1\ A\vec{v}_2\ \cdots\ A\vec{v}_n\,] = [\,\lambda_1\vec{v}_1\ \lambda_2\vec{v}_2\ \cdots \lambda_n\vec{v}_n\,] = \boldsymbol{[\,\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_n\,]\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}.$$

That is, $AP = PD$, where the columns of P are the eigenvectors of A. So if P has an inverse,

$$\boldsymbol{A = PDP^{-1}}, \text{ and } \boldsymbol{A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}}$$

and in general:

$$\boldsymbol{A^k = PD^kP^{-1}}.$$

But does P^{-1} always exist? Only if the columns of P are linearly independent, which is the case if P has a complete set of eigenvectors.

Recall that $A\vec{x} = \lambda\vec{x} \Rightarrow A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$, so A^{-1} has the same eigenvectors (the columns of P) as A with eigenvalues that are 1 / eigenvalues of A, so since $A = PDP^{-1}$, then

$$\boldsymbol{A^{-1} = PD^{-1}P^{-1} \text{ where } D^{-1} = \begin{bmatrix} 1/\lambda_1 & & & \\ & 1/\lambda_2 & & \\ & & \ddots & \\ & & & 1/\lambda_n \end{bmatrix}}.$$

This happens to be $A^k = PD^kP^{-1}$ for the case of $k = -1$.

We could also have found A^{-1} by finding $A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$.

Example 1: $A = \begin{bmatrix} -5 & 1 & 3 \\ -7 & 3 & 3 \\ -7 & 1 & 5 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix}$ with eigenvalues $-1, 2, 2$.

Then for this A we have $\boldsymbol{P} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{1} & \mathbf{7} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{7} \end{bmatrix}$ and $\boldsymbol{D} = \begin{bmatrix} \mathbf{-1} & & \\ & \mathbf{2} & \\ & & \mathbf{2} \end{bmatrix}$. Check that $PDP^{-1} = A$.

Then $$\boldsymbol{A}^{\mathbf{10}} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{1} & \mathbf{7} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{7} \end{bmatrix} \begin{bmatrix} (-\mathbf{1})^{\mathbf{10}} & & \\ & \mathbf{2}^{\mathbf{10}} & \\ & & \mathbf{2}^{\mathbf{10}} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{1} & \mathbf{7} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{7} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{-1363} & \mathbf{341} & \mathbf{1023} \\ \mathbf{-2387} & \mathbf{1365} & \mathbf{1023} \\ \mathbf{-2387} & \mathbf{341} & \mathbf{2047} \end{bmatrix}.$$

Example 2: $A = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with eigenvalues 1 , 0.92.

Then $A = PDP^{-1} = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .92 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}^{-1}$ and

$$\begin{aligned} \lim_{k\to\infty} \boldsymbol{A}^{k} &= \lim_{k\to\infty} \boldsymbol{P}\boldsymbol{D}^{k}\boldsymbol{P}^{-1} = \boldsymbol{P}\left(\lim_{k\to\infty} \boldsymbol{D}^{k}\right)\boldsymbol{P}^{-1} \\ &= \begin{bmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{5} & \mathbf{-1} \end{bmatrix} \left(\lim_{k\to\infty} \begin{bmatrix} \mathbf{1}^{k} & \mathbf{0} \\ \mathbf{0} & .\mathbf{92}^{k} \end{bmatrix} \right) \begin{bmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{5} & \mathbf{-1} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{5} & \mathbf{-1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{5} & \mathbf{-1} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} .\mathbf{375} & .\mathbf{375} \\ .\mathbf{675} & .\mathbf{675} \end{bmatrix}. \end{aligned}$$

If wanted, we can put eigenvectors in different forms/locations in diagonalizing a matrix.

Two more (of infinitely many) possible factorizations PDP^{-1} of $A = \begin{bmatrix} -5 & 1 & 3 \\ -7 & 3 & 3 \\ -7 & 1 & 5 \end{bmatrix}$ are with:

$\boldsymbol{D}$	$\boldsymbol{P}$
$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$	$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$
$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$	$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

Find an example of (create/build) a matrix with e-values , , and e-vectors $\begin{bmatrix} \\ \\ \end{bmatrix}, \begin{bmatrix} \\ \\ \end{bmatrix}, \begin{bmatrix} \\ \\ \end{bmatrix}$:

$$\boldsymbol{A} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}^{-1} \approx \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}.$$