Most important ideas:

- How to find eigenvalues of A using the characteristic equation $\det(A-\lambda I)=0$.
- Eigenvectors of $n\times n$ matrix A as a basis for R^n, and how this is extremely useful.

Recall: For square matrix A, if $A\vec{x}=\vec{0}$ for $\vec{x}\neq 0$ then A is "bad," including the fact that $\det A=0$. And the converse is true: $\det A=0 \Rightarrow A\vec{x}=\vec{0}$ for $\vec{x}\neq\vec{0}$. So to find the a value of λ for which $A\vec{x}=\lambda\vec{x}$, that is, $A\vec{x}-\lambda\vec{x}=\vec{0}$, that is, $(A-\lambda I)\vec{x}=\vec{0}$ for $\vec{x}\neq\vec{0}$, we find the values of λ for which $\det(A-\lambda I)=0$.

Example 1: Find the eigenvalues of $A=\begin{bmatrix}1&4\\3&2\end{bmatrix}$.

That is, find the values of λ for which $\det(A-\lambda I)=0$:

$$\boldsymbol{A-\lambda I=\begin{bmatrix}1&4\\3&2\end{bmatrix}-\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}1-\lambda&4\\3&2-\lambda\end{bmatrix},}$$

And

$$\boldsymbol{\begin{vmatrix}1-\lambda&4\\3&2-\lambda\end{vmatrix}=(1-\lambda)(2-\lambda)-3\cdot 4=\lambda^2-3\lambda-10=(\lambda+2)(\lambda-5)}$$

which $=0$ for $\lambda=-2$ or 5.

In Handout 5.1 we found corresponding eigenvectors: $\begin{bmatrix}-4\\3\end{bmatrix}$ for $\lambda=-2$ and $\begin{bmatrix}1\\1\end{bmatrix}$ for $\lambda=5$.

Example 2: Where $A=\begin{bmatrix}6&-2&0\\0&4&0\\1&-1&4\end{bmatrix}$, find value(s) for which $\det(A-\lambda I)=0$.

$$\boldsymbol{\begin{vmatrix}6-\lambda&-2&0\\0&4-\lambda&0\\1&-1&4-\lambda\end{vmatrix}=(4-\lambda)\begin{vmatrix}6-\lambda&0\\1&4-\lambda\end{vmatrix}=(4-\lambda)(4-\lambda)(6-\lambda)}$$

which $=0$ for $\lambda=4$ twice (that is, it has algebraic multiplicity of 2) and for $\lambda=6$.

In Handout 5.1 we found corresponding eigenvectors: $\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$ for $\lambda=4$ and $\begin{bmatrix}2\\0\\1\end{bmatrix}$ for $\lambda=6$.

Eigenvalue 4 has ***algebraic* multiplicity of 2** and **2 corresponding eigenvectors (so *geometric* multiplicity of 2)**.

Eigenvalue 6 has ***algebraic* multiplicity of 1** and **1 corresponding eigenvector (so *geometric* multiplicity of 1)**.

A matrix with real values can have complex (imaginary) eigenvalues and eigenvectors.

Example 3: Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

$$\begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 + 1 = \lambda^2 - 4\lambda + 5 = 0 \Rightarrow \lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} = 2 \pm i$$

We find the corresponding eigenvectors:

For $\lambda = 2 + i$, find the vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $A\vec{x} = (2+i)\vec{x}$, that is, $\big(A - (2+i)\big)\vec{x} = \vec{0}$:

$$\left[\begin{matrix} 2-(2+i) & -1 \\ 1 & 2-(2+i) \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right] \sim \left[\begin{matrix} -i & -1 \\ 1 & -i \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right] \underset{R2-R1}{\overset{i*R1}{\sim}} \left[\begin{matrix} 1 & -i \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ix_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Check that $\begin{bmatrix} i \\ 1 \end{bmatrix}$ is the correct eigenvector: $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 2i-1 \\ i+2 \end{bmatrix} = (2+i)\begin{bmatrix} i \\ 1 \end{bmatrix}$.

For $\lambda = 2 - i$, find the vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $A\vec{x} = (2-i)\vec{x}$, that is, $\big(A - (2-i)\big)\vec{x} = \vec{0}$:

$$\left[\begin{matrix} 2-(2-i) & -1 \\ 1 & 2-(2-i) \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right] \sim \left[\begin{matrix} i & -1 \\ 1 & i \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right] \underset{R2-R1}{\overset{-i*R1}{\sim}} \left[\begin{matrix} 1 & i \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -ix_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Check that $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ is the correct eigenvector: $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} -2i-1 \\ -i+2 \end{bmatrix} = (2-i)\begin{bmatrix} -i \\ 1 \end{bmatrix}$. ✓

A couple of notes about complex eigenvalues/vectors. As we just saw, a matrix with all real entries can have complex eigenvalues/vectors. This might seem a bit annoying. **However, complex eigenvalues have important real life meaning.** We'll see this in a few days.

Unfortunately, not every matrix has a complete (linearly independent) set of eigenvectors.

Example 4: Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$.

Eigenvalues: $\begin{vmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = 0 \Rightarrow \lambda = 2$ (twice).

Eigenvectors: find $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ so that $(A - 2I)\vec{x} = \vec{0}$: $\left[\begin{matrix} 2-2 & 3 \\ 0 & 2-2 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right] \sim \left[\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Check that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector: $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. ✓

It turns out that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the only eigenvector. We thought there might (should?) be 2, since the eigenvalue $\lambda = 2$ has algebraic multiplicity of 2 (it is a double eigenvalue).

An $n \times n$ matrix always has n eigenvalues, counting repeated roots of the characteristic equation $\det(A - \lambda I) = 0$.

As for the number of eigenvectors, all we can say that **for *each* eigenvalue of algebraic multiplicity k there is between 1 and k eigenvectors**.

We very much prefer than an $n \times n$ matrix has n different (i.e. linearly independent) eigenvectors, that is, a *complete* or *full* set of eigenvectors. ☺

The 2×2 matrix in Example 4 has only 1 eigenvector. ☹ We always hope that an $n \times n$ matrix has n different (linearly independent) eigenvectors.

It is extremely useful if the set of eigenvectors for an $n \times n$ matrix forms a basis for R^n.

Consider 2×2 matrix A with eigenvalues λ_1, λ_2 (possibly equal) and corresponding linearly independent eigenvectors $\vec{v}_1, \vec{v}_2$. That is, $A\vec{v}_1 = \lambda_1\vec{v}_1$ and $A\vec{v}_2 = \lambda_2\vec{v}_2$. Since there are two vectors and they are linearly independent, they form a basis for R^2. Let $\vec{x}$ be a vector from R^2. Then $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$ for some values c_1 and c_2. Then:

$$A^k\vec{x} = A^k(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1A^k\vec{v}_1 + c_2A^k\vec{v}_2 = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2 = \lambda_1^k\, c_1\vec{v}_1 + \lambda_2^k\, c_2\vec{v}_2 \,.$$

Example 5: $A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}$ has e-values $\lambda_1 = 1, \lambda_2 = 0.7$ and e-vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Check: $\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.7 \\ -0.7 \end{bmatrix} = 0.7\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. ✓ We see that the eigenvectors form a basis for R^2 since they are linearly independent. Suppose $\vec{x}_0 = \begin{bmatrix} \mathbf{7} \\ \mathbf{3} \end{bmatrix}$.

Then $\begin{bmatrix} \mathbf{7} \\ \mathbf{3} \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{7} \\ \mathbf{3} \end{bmatrix} = \frac{1}{-3}\begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} \mathbf{7} \\ \mathbf{3} \end{bmatrix} = \begin{bmatrix} \mathbf{10/3} \\ \mathbf{1/3} \end{bmatrix}$

so $\begin{bmatrix} \mathbf{7} \\ \mathbf{3} \end{bmatrix} = \frac{10}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \mathbf{20/3} \\ \mathbf{10/3} \end{bmatrix} + \begin{bmatrix} \mathbf{1/3} \\ \mathbf{-1/3} \end{bmatrix}$.

So we've broken the given vector $\begin{bmatrix} \mathbf{7} \\ \mathbf{3} \end{bmatrix}$ into two parts, each part an eigenvector of A.
Put another way, we can write the given vector as a linear combination of its eigenvectors.

Then:

$$\boldsymbol{A\vec{x}_0 = \lambda_1 c_1\vec{v}_1 + \lambda_2\, c_2\vec{v}_2 = 1\begin{bmatrix} 20/3 \\ 10/3 \end{bmatrix} + 0.7\begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}}$$

$$\boldsymbol{A^2\vec{x}_0 = 1^2\begin{bmatrix} 20/3 \\ 10/3 \end{bmatrix} + 0.7^2\begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}}$$

$$\boldsymbol{A^{10}\vec{x}_0 = 1^{10}\begin{bmatrix} 20/3 \\ 10/3 \end{bmatrix} + 0.7^{10}\begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}}$$

$$\boldsymbol{A^{\infty}\vec{x}_0 = \lim_{n\to\infty} A^n\vec{x}_0 = \quad A^2\vec{x}_0 = 1^{\infty}\begin{bmatrix} 20/3 \\ 10/3 \end{bmatrix} + 0.7^{\infty}\begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix} = 1\begin{bmatrix} 20/3 \\ 10/3 \end{bmatrix} + 0\begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 20/3 \\ 10/3 \end{bmatrix}}$$

As seen in Section 4.9, this is the equilibrium vector of A. Does every matrix have an equilibrium vector? No! Only those with an eigenvalue of 1 (so $A\vec{x} = 1\vec{x}$ for some $\vec{x}$) and with all other eigenvalues smaller than 1 in magnitude. This happens to be true of all probability/ *stochastic* matrices: matrices which have all ≥ 0 entries and for which each column's sum is 1.

The fact $A^k\vec{x} = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2$ is computationally helpful, but far more useful theoretically.

One final thought, an idea further discussed in Section 5.3, but which the book introduces in this section: Two matrices A and B are *similar* if $A = PBP^{-1}$ for some matrix P. Similar matrices have the same eigenvalues. See Theorem 4 and proof on page 277.