Most important ideas:

- Eigenvectors and eigenvalues.
- Various properties of eigenvectors and eigenvalues.
- Linear combinations of eigenvectors.
- Eigenspaces.

Example 1: "Random" matrix $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$. Then $A\vec{v} = \begin{bmatrix} 29 \\ 27 \end{bmatrix}$, which of course is not a multiple of $\vec{v}$. **In general, $A\vec{v} \neq$ some multiple of $\vec{v}$.**

So why are we even talking about this? Because for some matrices A, there are vectors $\vec{v}$ for which $A\vec{v} =$ some multiple of $\vec{v}$. In fact, for every matrix there is often one and sometimes many different such vectors, and **these vectors are extremely important and useful**.

Back to Example 1, with a more "carefully" chosen vector: $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \end{bmatrix} = -2\begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

Definition: If $A\vec{x} = \lambda\vec{x}$, then $\vec{x}$ is an eigen<u>vector</u> of A with corresponding eigen<u>value</u> λ.

Question: For a given matrix A, how to find these eigenvectors and eigenvalues?

First, note that $A\vec{x} = \lambda\vec{x}$ can also be written as $A\vec{x} - \lambda\vec{x} = \vec{0}$, that is, $(A - \lambda I)\vec{x} = \vec{0}$. So one way to find the vector $\vec{x}$ in $A\vec{x} = \lambda\vec{x}$ is to solve for $\vec{x}$ in $(A - \lambda I)\vec{x} = \vec{0}$.

(Notice that to do this we are assuming that we already know the value of eigenvalue λ. In Section 5.2 we will learn how to find the values of the eigenvalues λ for a given matrix.)

Example 1 again: Suppose we know that another eigenvalue of $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ is $\lambda = 5$. Let's find the corresponding eigenvector.

That is, let's find the vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for which $A\vec{x} = 5\vec{x}$, i.e. $(A - 5I)\vec{x} = \vec{0}$:

$A - 5I = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}$, and $\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 3 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$,

so $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so the vector $\vec{x}$ we are looking for is any multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Check: $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. ✓

So one eigenvalue of this matrix is $\lambda = 5$ with corresponding eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We also saw above that $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = -2\begin{bmatrix} -4 \\ 3 \end{bmatrix}$, so another eigenvalue of this matrix is $\lambda = -2$ with corresponding eigenvector $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

So just how many different eigenvalues and eigenvectors does a matrix have?

Example 2: $A = \begin{bmatrix} 6 & -2 & 0 \\ 0 & 4 & 0 \\ 1 & -1 & 4 \end{bmatrix}$ has eigenvalues $\lambda = 4$ and $\lambda = 6$. Find the eigenvectors.

For $\lambda = 4$: Solve for $\vec{x}$ in $A\vec{x} = 4\vec{x}$, that is, in $(A - 4I)\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

Eigenspace for $\lambda = 4$

So we just found two different (linearly independent) eigenvectors for the eigenvalue 4.

Check: $\begin{bmatrix} 6 & -2 & 0 \\ 0 & 4 & 0 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 6 & -2 & 0 \\ 0 & 4 & 0 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. ✓

Any linear combination of different eigenvectors corresponding to the <u>same</u> eigenvalue is also an eigenvector corresponding to that eigenvalue. More on this idea on the next page.

For $\lambda = 6$: Solve for $\vec{x}$ in $(A - 6I)\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Check: $\begin{bmatrix} 6 & -2 & 0 \\ 0 & 4 & 0 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. ✓

So this 3 x 3 matrix has three different eigenvectors (coincidence of 3?), two of which correspond to one eigenvalue, while a third eigenvector corresponds to a second eigenvalue.

Eigenvectors are non-zero vectors. Otherwise, every number would be an eigenvalue of every matrix A since $A\vec{0} = \lambda\vec{0}$ for any A and for any value of λ.

On a different note, if A has an eigenvalue of 0, then $A\vec{v} = 0\vec{v} = \vec{0}$ for some non-zero $\vec{v}$. That is, non-zero $\vec{v}$ is in the nullspace of A, which is bad, which also means that every other bad versions of the items in the IMT are also true.

Example 3: $A = \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix}$ has eigenvalues 0 and -2. What are the corresponding eigenvectors?

For $\lambda = -2$: Solve for $\vec{x}$ in $(A - (-2)I)\vec{x} = \vec{0}$: $\left[\begin{array}{cc|c} 3 & -3 & 0 \\ 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$,
so $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (and any multiple of it) is the eigenvector corresponding to eigenvalue $\lambda = -2$. Check: $\begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. ✓

For $\lambda = 0$: Solve for $\vec{x}$ in $(A - 0I)\vec{x} = \vec{0}$, i.e. in $A\vec{x} = \vec{0}$ $\left[\begin{array}{cc|c} 1 & -3 & 0 \\ 1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$,
so $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, so $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (and any multiple) is the eigenvector corresponding to eigenvalue $\lambda = -2$. Check: $\begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. ✓

We could already (i.e. before considering eigenvalues/vectors) see that A is bad since **the columns of A are linearly dependent, $\det A = 0$, etc.**

Four important properties. If A has the eigenvalue λ with two eigenvectors $\vec{u}$ and $\vec{v}$, that is, if $A\vec{u} = \lambda\vec{u}$ and $A\vec{v} = \lambda\vec{v}$, then the following are true:

1. $\boldsymbol{A(c\vec{v}) = c(A\vec{v}) = c(\lambda\vec{v}) = \lambda(c\vec{v})}$.

2. $\boldsymbol{A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \lambda\vec{u} + \lambda\vec{v} = \lambda(\vec{u}+\vec{v})}$.

3. $\boldsymbol{A^n\vec{v} = \lambda^n\vec{v}}$ **for** $\boldsymbol{n \geq 1}$. **This is actually true for any integer** $\boldsymbol{n}$**, whether positive, negative or 0.**

4. $\boldsymbol{A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}}$**, assuming that** $\boldsymbol{A^{-1}}$ **exists.**

Properties 1 and 2 in Example 2:

$$\begin{bmatrix} 6 & -2 & 0 \\ 0 & 4 & 0 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 6 & -2 & 0 \\ 0 & 4 & 0 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and where $\mathbf{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \mathbf{6}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{2} \\ \mathbf{2} \\ \mathbf{6} \end{bmatrix}$ we see that $\begin{bmatrix} 6 & -2 & 0 \\ 0 & 4 & 0 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} \mathbf{2} \\ \mathbf{2} \\ \mathbf{6} \end{bmatrix} = \begin{bmatrix} \mathbf{8} \\ \mathbf{8} \\ \mathbf{24} \end{bmatrix} = \mathbf{4}\begin{bmatrix} \mathbf{2} \\ \mathbf{2} \\ \mathbf{6} \end{bmatrix}$.

Warning: combining (as linear combinations) eigenvectors corresponding to *different* eigenvalues does not create an eigenvector. In Example 2:

$$\begin{bmatrix} 6 & -2 & 0 \\ 0 & 4 & 0 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 6 & -2 & 0 \\ 0 & 4 & 0 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

and where $\mathbf{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \mathbf{6}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{14} \\ \mathbf{2} \\ \mathbf{6} \end{bmatrix}$ we see that $\begin{bmatrix} 6 & -2 & 0 \\ 0 & 4 & 0 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} \mathbf{14} \\ \mathbf{2} \\ \mathbf{6} \end{bmatrix} = \begin{bmatrix} 80 \\ 8 \\ 36 \end{bmatrix}$ is not a multiple of $\begin{bmatrix} \mathbf{14} \\ \mathbf{2} \\ \mathbf{6} \end{bmatrix}$.

Property 3 in Example 2 (we use A^3):

$$\begin{bmatrix} 216 & -152 & 0 \\ 0 & 64 & 0 \\ 76 & -76 & 64 \end{bmatrix}\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{64} \\ \mathbf{64} \\ \mathbf{0} \end{bmatrix} = \mathbf{4^3}\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}.$$

$$\begin{bmatrix} 216 & -152 & 0 \\ 0 & 64 & 0 \\ 76 & -76 & 64 \end{bmatrix}\begin{bmatrix} \mathbf{2} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{432} \\ \mathbf{0} \\ \mathbf{216} \end{bmatrix} = \mathbf{6^3}\begin{bmatrix} \mathbf{2} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}.$$

Property 4 in Example 2 (so the matrix being used is A^{-1}):

$$\begin{bmatrix} 1/6 & 1/12 & 0 \\ 0 & 1/4 & 0 \\ -1/24 & 1/24 & 1/4 \end{bmatrix}\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/4 \\ 0 \end{bmatrix} = \mathbf{\frac{1}{4}}\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}.$$

$$\begin{bmatrix} 1/6 & 1/12 & 0 \\ 0 & 1/4 & 0 \\ -1/24 & 1/24 & 1/4 \end{bmatrix}\begin{bmatrix} \mathbf{2} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} 2/6 \\ 0 \\ 1/6 \end{bmatrix} = \mathbf{\frac{1}{6}}\begin{bmatrix} \mathbf{2} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}.$$

The eigenspace $\boldsymbol{\lambda(A)}$ **of a matrix** $\boldsymbol{A}$ **for a particular eigenvalue** $\boldsymbol{\lambda}$ **is the set of all eigenvectors corresponding to that eigenvalue, plus the zero vector.** The proof of this is given in Properties 1 and 2 above. We saw an example of an eigenspace at the top of page 2.

Feel free to use technology on larger problems to help with the work.