Most important ideas:

- **Markov Chains and equilibrium vectors.**
- **The idea that a matrix *operates* on (does something to) a vector to get a new vector.**

Example 1: Suppose that people live in either a city or its suburbs. Suppose that each year:

Of those who are currently in the city, 70% stay in the city and 30% move to the suburbs.

Of those who are currently in the suburbs, 50% move to the city and 50% stay in the suburbs.

Suppose that there are currently 1000 living in the city and 3000 living in the suburbs. How many people will there be in each a year from now? There will be:

$$(.70)1000 + (.50)3000 = 2200 \quad \text{living in the city}$$

$$(.30)1000 + (.50)3000 = 1800 \quad \text{living in the suburbs}$$

That is, after one year: $\vec{x}_1 = P\vec{x}_0 = \begin{bmatrix} .70 & .50 \\ .30 & .50 \end{bmatrix}\begin{bmatrix} 1000 \\ 3000 \end{bmatrix} = 1000\begin{bmatrix} .70 \\ .30 \end{bmatrix} + 3000\begin{bmatrix} .50 \\ .50 \end{bmatrix} = \begin{bmatrix} 2200 \\ 1800 \end{bmatrix}$.

After another year: $\vec{x}_2 = P\vec{x}_1 = \begin{bmatrix} .70 & .50 \\ .30 & .50 \end{bmatrix}\begin{bmatrix} 2200 \\ 1800 \end{bmatrix} = \begin{bmatrix} 2440 \\ 1560 \end{bmatrix}$. Note that $\vec{x}_2 = P(P\vec{x}_0) = P^2\vec{x}_0$.

After three years: $\vec{x}_3 = P\vec{x}_2 = \begin{bmatrix} .70 & .50 \\ .30 & .50 \end{bmatrix}\begin{bmatrix} 2440 \\ 1560 \end{bmatrix} \approx \begin{bmatrix} 2488 \\ 1512 \end{bmatrix}$. Note that $\vec{x}_3 = P(P(P\vec{x}_0)) = P^3\vec{x}_0$.

After many years: $\vec{x}_\infty = P^\infty\vec{x}_0 = \begin{bmatrix} 2500 \\ 1500 \end{bmatrix}$.

You could think of the proportions/fractions given above as probabilities. For example, there is a probability of .70 that someone living in the city will stay in the city and a probability of .30 of moving to the suburb. And similarly for those living in the suburbs.

P is a *probability* (a.k.a. *stochastic*) matrix: all of its entries are between 0 and 1 and its columns add up to 1.

Questions: is there a certain equilibrium population toward which the population is headed or will the population just fluctuate? And if there is an equilibrium population, how does the initial population affect what the equilibrium population will be, and how do the probabilities of moving to/from the city/suburbs affect that equilibrium population?

For probability (a.k.a. stochastic) matrix $P = \begin{bmatrix} .70 & .50 \\ .30 & .50 \end{bmatrix}$ (for example, using Excel) we find:

Initial population $\vec{x}_0$	$\begin{bmatrix} 1000 \\ 3000 \end{bmatrix}$	$\begin{bmatrix} 2000 \\ 6000 \end{bmatrix}$	$\begin{bmatrix} 2000 \\ 2000 \end{bmatrix}$	$\begin{bmatrix} 4000 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 4000 \end{bmatrix}$	$\begin{bmatrix} 5000 \\ -1000 \end{bmatrix}$	$\begin{bmatrix} .50 \\ .50 \end{bmatrix}$
Equilibrium population $\vec{x}_\infty = P^\infty\vec{x}_0$	$\begin{bmatrix} 2500 \\ 1500 \end{bmatrix}$	$\begin{bmatrix} 5000 \\ 3000 \end{bmatrix}$	$\begin{bmatrix} 2500 \\ 1500 \end{bmatrix}$	$\begin{bmatrix} 2500 \\ 1500 \end{bmatrix}$	$\begin{bmatrix} 2500 \\ 1500 \end{bmatrix}$	$\begin{bmatrix} 2500 \\ 1500 \end{bmatrix}$	$\begin{bmatrix} .625 \\ .375 \end{bmatrix}$

It seems that no matter what the initial population distribution is, **62.5%** of the final population will be in the city and **37.5%** in the suburbs.

What if we change the probabilities, say $P = \begin{bmatrix} .80 & .20 \\ .20 & .80 \end{bmatrix}$? Then (using Excel) we get:

Initial population $\vec{x}_0$	$\begin{bmatrix} 1000 \\ 3000 \end{bmatrix}$	$\begin{bmatrix} 2000 \\ 6000 \end{bmatrix}$	$\begin{bmatrix} 2000 \\ 2000 \end{bmatrix}$	$\begin{bmatrix} 4000 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 4000 \end{bmatrix}$	$\begin{bmatrix} 5000 \\ -1000 \end{bmatrix}$	$\begin{bmatrix} .50 \\ .50 \end{bmatrix}$
Equilibrium population $\vec{x}_\infty = P^\infty \vec{x}_0$	$\begin{bmatrix} 2000 \\ 2000 \end{bmatrix}$	$\begin{bmatrix} 4000 \\ 4000 \end{bmatrix}$	$\begin{bmatrix} 2000 \\ 2000 \end{bmatrix}$	$\left[\begin{bmatrix} 2000 \\ 2000 \end{bmatrix}\right]$	$\left[\begin{bmatrix} 2000 \\ 2000 \end{bmatrix}\right]$	$\left[\begin{bmatrix} 2000 \\ 2000 \end{bmatrix}\right]$	$\begin{bmatrix} .50 \\ .50 \end{bmatrix}$

So it looks like **the eventual proportions living in the city and suburbs of the equilibrium vector are affected by the values in the probability matrix (this is not a surprise, of course).**

Back to the original probability matrix from page 1. Notice $\begin{bmatrix} .70 & .50 \\ .30 & .50 \end{bmatrix}\begin{bmatrix} 2500 \\ 1500 \end{bmatrix} = \begin{bmatrix} 2500 \\ 1500 \end{bmatrix}$, giving us another way to view the equilibrium vector: it is the population $\vec{q}$ which after a year of transition (people moving to/from the city from/to the suburbs) doesn't change: $\boldsymbol{P\vec{q} = \vec{q}}$.

This is how we can find the equilibrium vector ↑ ↑ ↑

Example 2: Suppose $P = \begin{bmatrix} .70 & .50 \\ .30 & .50 \end{bmatrix}$. Then we want to find the values of c (for city) and s (for suburbs) which don't change after another year of transition, that is, find c and s so that

$$\begin{bmatrix} .70 & .50 \\ .30 & .50 \end{bmatrix}\begin{bmatrix} c \\ s \end{bmatrix} = \begin{bmatrix} c \\ s \end{bmatrix}, \text{ that is, } \begin{matrix} .70c + .50s = c \\ .30c + .50s = s \end{matrix}, \text{ that is, } \begin{matrix} -.30c + .50s = 0 \\ .30c - .50s = 0 \end{matrix}$$

whose augmented matrix we can row reduce to find a solution:

$$\left[\begin{array}{cc|c} -.3 & .5 & 0 \\ .3 & -.5 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -.3 & .5 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -5/3 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow c = \tfrac{5}{3}s, \text{ that is, } \begin{bmatrix} c \\ s \end{bmatrix} = \begin{bmatrix} \frac{5}{3}s \\ s \end{bmatrix} = s\begin{bmatrix} 5/3 \\ 1 \end{bmatrix}.$$

So any multiple of $\begin{bmatrix} 5/3 \\ 1 \end{bmatrix}$ **is an equilibrium vector, e.g.** $\begin{bmatrix} 1250 \\ 750 \end{bmatrix}$ **or** $\begin{bmatrix} 2500 \\ 1500 \end{bmatrix}$ **or** $\begin{bmatrix} 5/8 \\ 3/8 \end{bmatrix}$ **or** $\begin{bmatrix} .625 \\ .325 \end{bmatrix}$ **or any other vector in which the first value is 5/3 times the second value. The actual values depend on the initial total amounts: whatever the total initial population, regardless of how that initial total is divided between the city and the suburbs, after several years of moving between the city and suburbs, 5/8 of that initial total population will be in the city and 3/8 will be in the suburbs.** With that in mind, look again at the values at the bottom of the previous page.

Example 3: Suppose $P = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}$. We can find that $P^{\infty} = \begin{bmatrix} .375 & .375 \\ .625 & .625 \end{bmatrix}$.

In general, where $c + s = 1$ (so c is the fraction of the population in the city and s is the fraction of the population in the suburbs), we have

$$P^{\infty}\vec{q} = \begin{bmatrix} .375 & .375 \\ .625 & .625 \end{bmatrix}\begin{bmatrix} c \\ s \end{bmatrix} = c\begin{bmatrix} .375 \\ .625 \end{bmatrix} + s\begin{bmatrix} .375 \\ .625 \end{bmatrix} = (c+s)\begin{bmatrix} .375 \\ .625 \end{bmatrix} = \begin{bmatrix} .375 \\ .625 \end{bmatrix}.$$

Observation: Each column of P^{∞} is simply the equilibrium vector for that matrix.

This of course can occur for more complex situations with larger probability matrices (notice in the matrix below that the values in each column are between 0 and 1 and add up to 1).

For $P = \begin{bmatrix} .3355 & .3682 & .3067 & .0389 \\ .2663 & .2723 & .3277 & .5451 \\ .1935 & .1502 & .1589 & .2395 \\ .2047 & .2093 & .2067 & .1765 \end{bmatrix}$ we have $P^{\infty} = \begin{bmatrix} .2816 & .2816 & .2816 & .2816 \\ .3355 & .3355 & .3355 & .3355 \\ .1819 & .1819 & .1819 & .1819 \\ .2009 & .2009 & .2009 & .2009 \end{bmatrix}$.

Or we could have found the equilibrium vector by solving for $\vec{q}$ in $P\vec{q} = \vec{q}$, i.e. $(P - I)\vec{q} = \vec{0}$:

$$\left[\begin{array}{cccc|c} -.6645 & .3682 & .3067 & .0389 & 0 \\ .2663 & -.7277 & .3277 & .5451 & 0 \\ .1935 & .1502 & -.8411 & .2395 & 0 \\ .2047 & .2093 & .2067 & -.8235 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1.4016 & 0 \\ 0 & 1 & 0 & -1.6697 & 0 \\ 0 & 0 & 1 & -0.9053 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

So q is any multiple of $\begin{bmatrix} 1.4016 \\ 1.6697 \\ 0.9053 \\ 1.0000 \end{bmatrix}$, including $\frac{1}{1.4016+1.6697+0.9053+1.0000}\begin{bmatrix} 1.4016 \\ 1.6697 \\ 0.9053 \\ 1.0000 \end{bmatrix} = \begin{bmatrix} .2816 \\ .3355 \\ .1819 \\ .2009 \end{bmatrix}$.

In Chapter 5 we will look at finding vectors $\vec{v}$ that satisfy $P\vec{v} = \lambda\vec{v}$ for λ –values other than 1. Note that we are thinking of matrix P *acting* or *operating* $P\vec{v}$ *on* vector $\vec{v}$ to get a new vector.

See Numerical Note in Book Page 260.

There are all sorts of other applications in which probability/stochastic matrices and equilibrium vectors arise, such as consumers switching from one product to another (e.g. Coke vs. Pepsi) or voting (e.g. Republication vs. Democrat). You can bet that companies, political parties, etc. are willing to pay top dollar to someone who can understand the mathematics of all of this and help them predict and perhaps even manipulate the future.