Most important ideas:

- **Column space, row space; rank, nullity.**

First a review of some old ideas, using $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

The column space of A consists of all vectors that can be built as linear combinations $c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ using the columns of A. That is, $Col\ A = span\left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}\right\}$.

$\left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}\right\}$ is linearly dependent because: **there are too many vectors** —see Theorem 9, page 225.

We know that, as a vector space, $Col\ A$ has a basis consisting of one or more vectors from $\left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}\right\}$—see Thm. 5(b), page 210. We can see that $Col\ A = span\left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}\right\} = R^2$, so any two of the three columns of A form a basis for $Col\ A$, as long as those columns are linearly independent. For example $\left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}\right\}$ forms a basis for $Col\ A$.

Now for some new ideas.

The **row space** of A consists of all of the rows (i.e. "row vectors") that can built from the rows of A, that is, $Row\ A = span\{[1 \quad 2 \quad 3], [4 \quad 5 \quad 6]\}$, which is sort of the same as the column space of $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$, where $Col\ A^T = span\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right\}$.

The **rank** of A is the dimension of the column space of A.

Note: $\dim Col\ A =$ number of linearly independent columns in A

$=$ number of pivot columns

$=$ number of pivot rows

$=$ number of linearly independent rows

$= \dim Row\ A$.

So $rank\ A = \dim Col\ A = dim\ Row\ A =$ number of pivots in A. The rank essentially tells us how "good" a matrix is. A matrix is of **full column rank** if it has pivots in every column, of **full row rank** if it has pivots in every row, and simply of **full rank** if it has pivots in every column and row (which would mean that the matrix is square and invertible). We prefer a full rank matrix.

Example: 3×5 matrix $A = \begin{bmatrix} 1 & 3 & 4 & 3 & 11 \\ -1 & -3 & 1 & 2 & -1 \\ 0 & 0 & -2 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

One basis for $Col\ A$ is $\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}\right\}$, and $\mathbf{\dim Col\ A = 2.}$

One basis for $Row\ A$ is $\{[1 \quad 3 \quad 0 \quad -1 \quad 3], [0 \quad 0 \quad 1 \quad 1 \quad 2]\}$, and $\mathbf{\dim Row\ A = 2.}$

So $\mathbf{rank\ A = \dim Col\ A = \dim Row\ A = 2}$ (not full rank, so A is bad in some way).

Also notice that there are **3** non-pivot columns in A which means there are **3** free variables in $A\vec{x} = \vec{b}$, assuming there is a solution at all. In particular, the solution to $A\vec{x} = \vec{0}$ has **3** free variables. That is, the dimension of the nullspace $Nul\ A$ of A is **3**.

Let's find the nullspace of A above. We have already row-reduced A to get:

$$\begin{bmatrix} 1 & 3 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ which means } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_2 + x_4 - 3x_5 \\ x_2 \\ -x_4 - 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

from which we can see that $\mathbf{\dim \mathit{Nul}\, \mathit{A} = 3}$. (Note: each of the three vectors is in the nullspace of A and so is any linear combination of them.)

Observations:

First, the dimension of the column space is at most the number of columns (it could be less, if any of the columns are linear combinations of the other columns), and the dimension of the row space is at most the number of rows (it could also be less, if any of the rows are linear combinations of the other rows).

Second, since $rank\ A = \dim Col\ A = \dim Row\ A$. Then for $m \times n$ matrix A, we have

$$rank\ A = \dim Col\ A \le m$$

$$rank\ A = \dim Row\ A \le n$$

which together mean

$$\boldsymbol{rank\ A \le \min(m, n)}$$

Recall that the rank of the matrix is the number of pivots in the (reduced or not) row echelon form of the matrix, and thus must be $\le$ the number of rows and $\le$ the number of columns.

Note: $\boldsymbol{rank\ A^T = \dim Col\ A^T = \dim Row\ A = \dim Col\ A = rank\ A}$.

Just as $\boldsymbol{rank\ A = \dim Col\ A}$, we define $\boldsymbol{nullity\ A = \dim Nul\ A}$.

Recall that $m \times n$ matrix A has n columns, so of course:

the number of pivot columns + the number of non-pivot columns $\boldsymbol{= n}$

That is:

$$\boldsymbol{rank\ A + nullity\ A = n}$$

In the previous example: $\boldsymbol{rank\ A = 2,\ nullity\ A = 3,\ n = 5}$

See the Invertible Matrix Theorem items (m) through (r).

For an $n \times n$ matrix we hope that $\boldsymbol{rank\ A = n}$ (which means all sorts of other good things) and we hope that $\boldsymbol{nullity\ A = 0}$, which means the only solution to $A\vec{x} = \vec{0}$ is $\boldsymbol{\vec{x} = \vec{0}}$.

To be clear, the pivots columns **in A itself** form a basis for $Col\ A$ while the pivot rows in the **(reduced or not) echelon form of A** form a basis for $Row\ A$. Actually the pivot rows in **A itself** also form a basis for $Row\ A$, but the pivot rows in the the reduced row echelon form of A are simpler and thus easier to work with.

In the above example, with $A = \begin{bmatrix} 1 & 3 & 4 & 3 & 11 \\ -1 & -3 & 1 & 2 & -1 \\ 0 & 0 & -2 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$:

A basis for $Col\ A$ is: $\left\{ \begin{bmatrix} \mathbf{1} \\ \mathbf{-1} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{4} \\ \mathbf{1} \\ \mathbf{-2} \end{bmatrix} \right\}$

A basis for $Col\ A$ is not: $\left\{ \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix} \right\}$

A basis for $Row\ A$ is: $\{[\mathbf{1} \quad \mathbf{3} \quad \mathbf{4} \quad \mathbf{3} \quad \mathbf{11}], [\mathbf{-1} \quad \mathbf{-3} \quad \mathbf{1} \quad \mathbf{2} \quad \mathbf{-1}]\}$

Another basis for $Row\ A$ is: $\{[\mathbf{1} \quad \mathbf{3} \quad \mathbf{0} \quad \mathbf{-1} \quad \mathbf{3}], [\mathbf{0} \quad \mathbf{0} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{2}]\}$

Reminder: $rank\ A$ is simply the number of pivots in a matrix, and the more the better. Let's compare and contrast possible values of $rank\ A$ and $nullity\ A$ and what it all means:

Columns of A

Dimension of A	If $rank\ A$ were…	…then $nullity\ A$ would be	Linearly independent?	Span R^m?	Comments:
3×5	3	**2**	**No**	**Yes**	
	2	**3**	**No**	**No**	
	1	**4**	**No**	**No**	
	0	**5**	**No**	**No**	**A is an all zero matrix**
3×3	3	**0**	**Yes**	**Yes**	**A is an invertible matrix**
	2	**1**	**No**	**No**	
	1	**2**	**No**	**No**	
	0	**3**	**No**	**No**	**A is an all zero matrix**
5×3	3	**0**	**Yes**	**No**	
	2	**1**	**No**	**No**	
	1	**2**	**No**	**No**	
	0	**3**	**No**	**No**	**A is an all zero matrix**

Observations: The columns of a 3×5 matrix cannot be linearly independent (too many)

The columns of a 5×3 matrix cannot span R^5 (not enough)