Most important ideas:

- **The dimension of a vector space is equal to the number of vectors in its basis.**
- **The Basis Theorem on page 227.**

Example 1: Consider $\left\{\begin{bmatrix}7\\3\\2\end{bmatrix},\begin{bmatrix}2\\6\\5\end{bmatrix},\begin{bmatrix}1\\9\\0\end{bmatrix},\begin{bmatrix}5\\-15\\-21\end{bmatrix}\right\}$ where $\begin{bmatrix}7&2&1&5\\3&6&9&-15\\2&5&0&-21\end{bmatrix}\sim\begin{bmatrix}1&0&0&2\\0&1&0&-5\\0&0&1&1\end{bmatrix}$.

Are the columns linearly independent? **No, there is not a pivot in every column.**
We knew this would be the case, since the number of vectors > size of each vector.

Do the columns span R^3? **Yes, there is a pivot in every row.**

Theorem 9, page 225: If a vector space has dimension n, then any set of vectors with more than n vectors must be **linearly dependent** (although fewer than n vectors doesn't guarantee linear independence). For example, four vectors from R^3 are necessarily linearly dependent.

Example 2: A basis for R^3 is: Another basis for R^3 is: Another basis for R^3 is:

$$\left\{\begin{bmatrix}7\\3\\2\end{bmatrix},\begin{bmatrix}2\\6\\5\end{bmatrix},\begin{bmatrix}1\\9\\0\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{bmatrix}7\\8\\0\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

Every basis for R^3 has **3** vectors, so the dimension of R^3 is **3**. See Theorem 10, page 226.

The dimension of the vector space which is simply $\{\vec{0}\}$ is defined to be **0.**

Example 3: Suppose a basis for R^n (for some n) is $\{\vec{v}_1,\vec{v}_2,\dots,\vec{v}_7\}$. What is n? **7.**

Theorem 1, page 227. If V is a p–dimensional vector space ($p \geq 1$), any set of p vectors from V that…

…is linearly independent <u>also</u> spans V (and thus is a basis for V).

…spans V <u>also</u> is linearly independent (and thus is a basis for V).

We saw this idea in 4.3: see your class handout, near the bottom of the page 1.

Example 4: Find a basis for the column space of the 4×5 matrix A:

$$A=\begin{bmatrix}1&-2&1&1&5\\2&-4&0&1&10\\3&-6&0&1&13\\4&-8&1&1&14\end{bmatrix}\sim\begin{bmatrix}\mathbf{1}&\mathbf{-2}&\mathbf{0}&\mathbf{0}&\mathbf{3}\\\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{-2}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{4}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{bmatrix}.$$

So a basis is $\left\{\begin{bmatrix}\mathbf{1}\\\mathbf{2}\\\mathbf{3}\\\mathbf{4}\end{bmatrix},\begin{bmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\\\mathbf{1}\end{bmatrix},\begin{bmatrix}\mathbf{1}\\\mathbf{1}\\\mathbf{1}\\\mathbf{1}\end{bmatrix}\right\}$, and $\dim Col\ A = \mathbf{3}$ (the number of pivot columns).

Reminder: the non-pivot columns tells us how the extra two vectors (that are not part of the basis) can be built out of the three pivot columns found in the basis:

$$\begin{bmatrix}\mathbf{-2}\\\mathbf{-4}\\\mathbf{-6}\\\mathbf{-8}\end{bmatrix}=\mathbf{-2}\begin{bmatrix}\mathbf{1}\\\mathbf{2}\\\mathbf{3}\\\mathbf{4}\end{bmatrix}+\mathbf{0}\begin{bmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\\\mathbf{1}\end{bmatrix}+\mathbf{0}\begin{bmatrix}\mathbf{1}\\\mathbf{1}\\\mathbf{1}\\\mathbf{1}\end{bmatrix} \quad\text{and}\quad \begin{bmatrix}\mathbf{5}\\\mathbf{10}\\\mathbf{13}\\\mathbf{14}\end{bmatrix}=\mathbf{3}\begin{bmatrix}\mathbf{1}\\\mathbf{2}\\\mathbf{3}\\\mathbf{4}\end{bmatrix}-\mathbf{2}\begin{bmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\\\mathbf{1}\end{bmatrix}+\mathbf{4}\begin{bmatrix}\mathbf{1}\\\mathbf{1}\\\mathbf{1}\\\mathbf{1}\end{bmatrix}$$

As we've seen, to find $Nul\ A$ we need to do some work. Using the row reduced matrix we just found above, the solution to the homogenous problem (i.e., with right hand side of $\vec{0}$) is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 - 3x_5 \\ x_2 \\ 2x_5 \\ -4x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ -4 \\ 1 \end{bmatrix}, \text{ so } Nul\ A = span\{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ -4 \\ 1 \end{bmatrix}\}.$$

We just discovered (or were reminded) that the dimension of $Col\ A$ is equal to the number of pivot columns (since the pivot columns of a matrix tell you which of its columns form a basis for its column space) and the dimension of $Nul\ A$ is equal to the number of non-pivot columns (since the non-pivot columns tell you which of the variables of $\vec{x}$ are free in finding the solution to $A\vec{x} = \vec{0}$).

Notice: **number of pivot columns + number of non-pivot columns = total number of columns**

In general: for $m \times n$ matrix A: $\mathbf{\dim} \boldsymbol{Col\ A} + \mathbf{\dim} \boldsymbol{Nul\ A} = \boldsymbol{n}$

Also see Book Example 5, page 228.

Note: Suppose H is a subspace of V, then $\dim H \leq \dim V$. See Theorem 11 on page 227.

Example 5: $V = R^3$ (which has dimension of 3), $H = \{\begin{bmatrix} a+b \\ -a+b \\ a \end{bmatrix} : a, b \in R\}$. So $H \subseteq V$.

So H consists of all vectors of the form $\begin{bmatrix} a+b \\ -a+b \\ a \end{bmatrix} = \boldsymbol{a}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \boldsymbol{b}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, so $H = span\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\}$,

which is a basis for H, since the vectors are **linearly independent**. So $\mathbf{\dim} \boldsymbol{H} = \mathbf{2}$.

And we see that $\dim H \leq dim\ V$ $(2 \leq 3)$.

Why not just put $<$ rather than $\leq$? **The two dimensions can be equal, as seen in the next example.**

Example 6: $V = R^2$ (so $\mathbf{\dim} \boldsymbol{V} = \mathbf{2}$), $H = Col\ A$ where $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$, so $H \subseteq V$.

$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$, so **a basis for $\boldsymbol{Col\ A}$ is $\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\}$**, so $\mathbf{\dim} \boldsymbol{H} = \mathbf{2}$ (and of course $2 \leq 2$).

Example 7: Suppose that V is the set of all 2×2 matrices, and H is the set of all real symmetric (that is, $A^T = A$) 2×2 matrices; that is, $H = \{\begin{bmatrix} a & b \\ b & c \end{bmatrix} : a, b, c \in R\}$.

Then elements of (the things in) H are of the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,

so a basis for H is $\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$. So $\mathbf{\dim} \boldsymbol{H} = \mathbf{3}$.

What is $\dim V$? $\mathbf{\dim} \boldsymbol{V} = \mathbf{4}$, since in general you get to choose **4** values for each matrix in V.

And of course $\dim H \leq \dim V$, since $H \subseteq V$.

Some examples of how $\dim H$ and $\dim T(H)$ are related and how the matrix A, where $T(\vec{v}) = A\vec{v}$, affects this.

	$H = span\{\vec{v}_1, \vec{v}_2\}$ for $\vec{v}_1, \vec{v}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}$ $\dim H = 2$			$H = span\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for $\vec{v}_1, \vec{v}_2, \vec{v}_3 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ $\dim H = 3$			
$T(\vec{v}) = A\vec{v}$ for $A =$	$T(\vec{v}_1), T(\vec{v}_2)$	$T: R^3 \to R^?$	$\dim T(H)$	$T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)$	Reduced echelon form of $[\,T(\vec{v}_1)\ \ T(\vec{v}_2)\ \ T(\vec{v}_3)]$	$T: R^3 \to R^?$	$\dim T(H)$
$\begin{bmatrix}1&2&3\\4&5&6\\7&8&0\end{bmatrix}$	$\begin{bmatrix}4\\10\\7\end{bmatrix}, \begin{bmatrix}3\\9\\15\end{bmatrix}$	$T: R^3 \to R^{3}$	**2**	$\begin{bmatrix}4\\10\\7\end{bmatrix}, \begin{bmatrix}3\\9\\15\end{bmatrix}, \begin{bmatrix}6\\15\\15\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	$T: R^3 \to R^{3}$	**3**
$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$	$\begin{bmatrix}4\\10\\16\end{bmatrix}, \begin{bmatrix}3\\9\\15\end{bmatrix}$	$T: R^3 \to R^{3}$	**2**	$\begin{bmatrix}4\\10\\16\end{bmatrix}, \begin{bmatrix}3\\9\\15\end{bmatrix}, \begin{bmatrix}6\\15\\24\end{bmatrix}$	$\begin{bmatrix}1&0&3/2\\0&1&0\\0&0&0\end{bmatrix}$	$T: R^3 \to R^{3}$	**2**
$\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$	$\begin{bmatrix}4\\10\end{bmatrix}, \begin{bmatrix}3\\9\end{bmatrix}$	$T: R^3 \to R^{2}$	**2**	$\begin{bmatrix}4\\10\end{bmatrix}, \begin{bmatrix}3\\9\end{bmatrix}, \begin{bmatrix}6\\15\end{bmatrix}$	$\begin{bmatrix}1&0&3/2\\0&1&0\end{bmatrix}$	$T: R^3 \to R^{2}$	**2**
$\begin{bmatrix}1&2&3\\4&8&12\end{bmatrix}$	$\begin{bmatrix}4\\16\end{bmatrix}, \begin{bmatrix}3\\12\end{bmatrix}$	$T: R^3 \to R^{2}$	**1**	$\begin{bmatrix}4\\16\end{bmatrix}, \begin{bmatrix}3\\12\end{bmatrix}, \begin{bmatrix}6\\24\end{bmatrix}$	$\begin{bmatrix}4&3&6\\0&0&0\end{bmatrix}$	$T: R^3 \to R^{2}$	**1**
$\begin{bmatrix}1&2&3\\4&5&6\\7&8&0\\1&0&1\end{bmatrix}$	$\begin{bmatrix}4\\10\\7\\2\end{bmatrix}, \begin{bmatrix}3\\12\\15\\1\end{bmatrix}$	$T: R^3 \to R^{4}$	**2**	$\begin{bmatrix}4\\10\\7\\2\end{bmatrix}, \begin{bmatrix}3\\12\\15\\1\end{bmatrix}, \begin{bmatrix}6\\15\\15\\2\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$	$T: R^3 \to R^{4}$	**3**
$\begin{bmatrix}1&2&3\\4&8&12\\7&14&21\\3&6&9\end{bmatrix}$	$\begin{bmatrix}4\\16\\28\\12\end{bmatrix}, \begin{bmatrix}3\\12\\21\\9\end{bmatrix}$	$T: R^3 \to R^{4}$	**1**	$\begin{bmatrix}4\\16\\28\\12\end{bmatrix}, \begin{bmatrix}3\\12\\21\\9\end{bmatrix}, \begin{bmatrix}6\\24\\42\\18\end{bmatrix}$	$\begin{bmatrix}1&3/4&6/4\\0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$T: R^3 \to R^{4}$	**1**

So depending on A in $T(\vec{v}) = A\vec{v}$, $\dim T(H)$ may be smaller than $\dim H$, but can never be larger. That is, $\dim T(H) \leq \dim H$.

It turns out that $\dim T(H) = \dim H$ is only possible if the columns of A are linearly independent (which also means T is a 1-to-1 function).