Most important ideas:

- **You can build (as a linear combination) one vector out of others. The coordinates of a vector with respect to a particular basis are the amounts of each vector in the basis needed to build the one vector.**

Examples:

Basis B	Amount of each vector needed to build $\begin{bmatrix}1\\2\end{bmatrix}$	Coordinates of $\begin{bmatrix}1\\2\end{bmatrix}$ with respect to the basis
$\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}$	$\mathbf{1}\begin{bmatrix}\mathbf{1}\\\mathbf{0}\end{bmatrix}+\mathbf{2}\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}$	$(\mathbf{1},\mathbf{2})=\begin{bmatrix}\mathbf{1}\\\mathbf{2}\end{bmatrix}$
$\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}3\\4\end{bmatrix}$	$\mathbf{1}\begin{bmatrix}\mathbf{1}\\\mathbf{2}\end{bmatrix}+\mathbf{0}\begin{bmatrix}\mathbf{3}\\\mathbf{4}\end{bmatrix}$	$(\mathbf{1},\mathbf{0})=\begin{bmatrix}\mathbf{1}\\\mathbf{0}\end{bmatrix}$
$\begin{bmatrix}5\\6\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}$	$\mathbf{1}\begin{bmatrix}\mathbf{5}\\\mathbf{6}\end{bmatrix}-\mathbf{4}\begin{bmatrix}\mathbf{1}\\\mathbf{1}\end{bmatrix}$	$(\mathbf{1},-\mathbf{4})=\begin{bmatrix}\mathbf{1}\\-\mathbf{4}\end{bmatrix}$

How do we find these coordinates? Let's look at the third case:

$$c_1\begin{bmatrix}5\\6\end{bmatrix}+c_2\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}\Rightarrow\begin{bmatrix}5&1\\6&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}\Rightarrow\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}5&1\\6&1\end{bmatrix}^{-1}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1\\-4\end{bmatrix}.$$

How do we know that inverse exists?

Its columns are linearly independent, its determinant is $\neq 0$, etc.

In general, where $B=\{\vec{b}_1,\vec{b}_2,\dots,\vec{b}_n\}$ is a basis for R^n, then any $\vec{x}\in R^n$ can be written

$$\vec{x}=c_1\vec{b}_1+c_2\vec{b}_2+\cdots+c_n\vec{b}_n=\left[\,\vec{b}_1\,\vec{b}_2\cdots\vec{b}_n\,\right]\begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix}\text{ where }\begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix}=\left[\,\vec{b}_1\,\vec{b}_2\cdots\vec{b}_n\,\right]^{-1}\vec{x},$$

as we just saw in the previous example.

We say that <u>**the coordinates $[\vec{x}]_B$ of $\vec{x}$ with respect to B**</u> are $(c_1,c_2,\dots,c_n)$.

In the above, P_B is called the <u>projection matrix</u> or <u>change-of-basis</u> matrix.

Example: Where $B=\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ and $\vec{x}=\begin{bmatrix}2\\5\\4\end{bmatrix}$, the coordinates of $\vec{x}$ with respect to B are

$$[\vec{x}]_B=\begin{bmatrix}1&1&1\\2&0&1\\3&1&1\end{bmatrix}^{-1}\begin{bmatrix}2\\5\\4\end{bmatrix}=\begin{bmatrix}-1/2&0&1/2\\1/2&-1&1/2\\1&1&-1\end{bmatrix}\begin{bmatrix}2\\5\\4\end{bmatrix}=\begin{bmatrix}1\\-2\\3\end{bmatrix}.$$

That is, $\begin{bmatrix}2\\5\\4\end{bmatrix}=1\begin{bmatrix}1\\2\\3\end{bmatrix}+(-2)\begin{bmatrix}1\\0\\1\end{bmatrix}+3\begin{bmatrix}1\\1\\1\end{bmatrix}.$

Example: Where $B = \{1, 1+t, 1+t+t^2\}$ and $p(t) = 4 + 5t + 2t^2$, then

$$p(t) = 4 + 5t + 2t^2 = c_1(1) + c_2(1+t) + c_3(1+t+t^2)$$

means

$$\begin{aligned} c_1 + c_2 + c_3 &= 4 \\ c_2 + c_3 &= 5 \\ c_3 &= 2 \end{aligned} \qquad \text{that is, } \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}.$$

We are reminded that we can think of a polynomial as a vector:

Polynomials 1, $1+t$, $1+t+t^2$ and $4+5t+2t^2$ correspond to vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix}$.

So the coordinates of $p(t)$ with respect to B are

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = [p(t)]_B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}.$$

That is, $\begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$,

that is, $\mathbf{4 + 5t + 2t^2 = (-1)(1) + 3(1+t) + 2(1+t+t^2)}$.

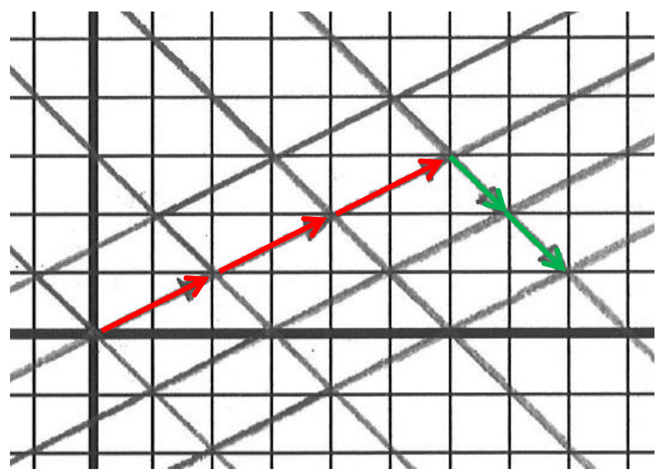
Example: Suppose $B = \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$, $\vec{x} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$.

Then $[\vec{x}]_B = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

So $\begin{bmatrix} 8 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. What does this look like? (It's sort of like we are changing the axes.) Also see Figures 1 and 2 on page 217 for a similar scenario.



Theorem 8 on page 219 simply says that **for each $\vec{x}$ there is *exactly* one way (that is, there is *one and only one way*) of building $\vec{x}$ out of the vectors in a given basis.** If $B = \{\vec{b}_1, \ldots, \vec{b}_n\}$ (so the projection matrix is $P_B = [\,\vec{b}_1 \cdots \vec{b}_n\,]$) and for a vector $\vec{x}$, we have

$$\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n = [\,\vec{b}_1 \cdots \vec{b}_n\,] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = P_B[\vec{x}]_B$$

so that $[\vec{x}]_B = P_B^{-1}\vec{x}$, which is the one and only one way of building $\vec{x}$ out of vectors $\vec{b}_1, \ldots, \vec{b}_n$.

Miscellaneous note: a function that is a one-to-one is sometimes called an <u>isomorphism</u>. For example, P_3, the set of third degree polynomials is isomorphic to R^4:

Each polynomial $a_0 + a_1 t + a_2 t^2 + a_3 t^3$ corresponds to a vector $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$, and conversely.