This porridge is too hot, this porridge it too cold, this porridge is just right. —Goldilocks

Most important ideas:

- **Basis: a collection of vectors that span the vector space (enough) and that are linearly independent (not too many).**
- **Every vector space has a basis (actually an infinite number of bases), and each basis has the same number of vectors: the dimension of the vector space.**
- **We can think of polynomials as vectors.**

Reminder: a set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is linearly <u>in</u>dependent if:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0} \Rightarrow c_1 = c_2 = \cdots = c_n = 0, \text{ that is, } [\,\vec{v}_1\;\vec{v}_2 \cdots \vec{v}_n\,]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

And set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is linearly <u>de</u>pendent means (Theorem 4 on page 208):
At least one vector $\vec{v}_i$ can be built using (is a linear combination of) the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}$.

If these vectors from R^m are linearly independent, then it must be that $n \le m$. We can show this using pivot columns: for the columns to be linearly independent, it must be that we have a pivot in every **<u>column</u>**, which also means we need **# columns $\le$ # rows.** A few examples (the reduced echelon forms) of columns in a matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$$

Independent **Independent** **Dependent** **Dependent**

Reminder: a set of n vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ from R^m spans R^m if **every vector in R^m can be built (as a linear combination) using $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$, that is, for each $\vec{b}$ in R^n, there are values $x_1, x_2, \dots, x_n$ so that $\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$, that is, $\vec{b} = [\,\vec{v}_1\;\vec{v}_2 \cdots \vec{v}_n\,]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.**

If these vectors from R^m span R^m, then it must be that $n \ge m$. We can show this in terms of pivot rows: for the columns to span R^m, it must be that we have a pivot in every **<u>row</u>**, which means we need **# columns $\ge$ # rows.**

If a set of vectors from R^m (1) spans R^m (so we have "enough" vectors) and (2) is linearly independent (so we don't have "too many" vectors), then we say that that set of vectors is a **<u>basis</u>** for R^m. See Definition on page 209. If this is the case, then as we saw above, we have both $n \le m$ and $n \ge m$, which means that $n = m$. For example, a basis for R^3 would have exactly **3** vectors. However, having the right number of vectors doesn't necessarily guarantee it is a basis. (We would say that having the right number of vectors is a *necessary* but not a *sufficient* condition.)

Examples: are the following bases for R^3? Note: "bases" is plural of "basis."

Vectors **and corresponding matrix reduced row echelon form**	Columns lin. indep.? (Pivot in every column?)	Columns span R^3? (Pivot in every row?)	Columns basis for R^3?
$\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{bmatrix}7\\8\\0\end{bmatrix}\right\}$ $\begin{bmatrix}1&4&7\\2&5&8\\3&6&0\end{bmatrix}\sim\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	**Yes**	**Yes**	**Yes**
$\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{bmatrix}7\\8\\9\end{bmatrix}\right\}$ $\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}\sim\begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\end{bmatrix}$	**No**	**No**	**No**
$\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix}\right\}$ $\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}\sim\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$	**Yes**	**No**	**No**
Any two vectors from R^3 $\left\{\begin{bmatrix}\ \\ \ \\ \ \end{bmatrix},\begin{bmatrix}\ \\ \ \\ \ \end{bmatrix}\right\}$	**Don't know—depends on actual values.**	**No**	**No**
$\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{bmatrix}7\\8\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ $\begin{bmatrix}1&4&7&1\\2&5&8&1\\3&6&0&1\end{bmatrix}\sim\begin{bmatrix}1&0&0&-1/3\\0&1&0&1/3\\0&0&1&0\end{bmatrix}$	**No**	**Yes**	**No**
Any four vectors from R^4 $\left\{\begin{bmatrix}\ \\ \ \\ \ \\ \ \end{bmatrix},\begin{bmatrix}\ \\ \ \\ \ \\ \ \end{bmatrix},\begin{bmatrix}\ \\ \ \\ \ \\ \ \end{bmatrix},\begin{bmatrix}\ \\ \ \\ \ \\ \ \end{bmatrix}\right\}$	**No**	**Don't know—depends on actual values.**	**No**

The <u>standard basis</u> for R^3 is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$, and similarly for any R^n.

Really useful fact:
For vectors $\vec{v}_1, \dots, \vec{v}_n \in R^m$, any two being true guarantees that the third is also true:

(1) $m = n$

(2) $\{\vec{v}_1, \dots, \vec{v}_n\}$ **is linearly independent**

(3) $\{\vec{v}_1, \dots, \vec{v}_n\}$ **span** R^m

Also look at the Invertible Matrix Theorem items (e) and (h).

Every vector space has a basis. See Theorem 5 on page 210. You'd kind of assume this is true, but as mathematicians we prove everything, just to be safe.

The column space of A is the collection of all vectors that can be built (as linear combinations of) from the columns of A, so of course the columns of A span the column space of A. But it may be that there are some "extra" columns in A, that is, columns of A which are built from (linear combinations of) other columns of A. So which columns of A do we keep and which can we get rid of to be more efficient? That is, which columns of A form a basis for $Col\ A$?

Answer: **Pivot columns**

What a surprise (NOT!) that **pivots** once again give us useful information about the matrix.

Example: $A = \begin{bmatrix} 1 & -2 & 7 & 11 & 4 \\ 2 & -4 & 8 & 12 & 4 \\ 3 & -6 & 9 & 13 & 4 \\ 4 & -8 & 10 & -11 & -21 \end{bmatrix} \sim \begin{bmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$

So a basis for the column space of A is $\left\{ \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{bmatrix}, \begin{bmatrix} \mathbf{7} \\ \mathbf{8} \\ \mathbf{9} \\ \mathbf{10} \end{bmatrix}, \begin{bmatrix} \mathbf{11} \\ \mathbf{12} \\ \mathbf{13} \\ \mathbf{-11} \end{bmatrix} \right\}$**.**

(However, these columns do not form a basis for $\boldsymbol{R^4}$.)

Example: Is $\{1, 2+3t, 4+5t+6t^2\}$ a basis for P_2, the set of polynomials $p(t) = a_0 + a_1 t + a_2 t^2$?

First, does $\boldsymbol{\{1, 2+3t, 4+5t+6t^2\}}$ span $\boldsymbol{P_2}$?

That is, can any $\boldsymbol{p(t) = a_0 + a_1 t + a_2 t^2}$ be written as a linear comb. of $\boldsymbol{\{1, 2+3t, 4+5t+6t^2\}}$?

That is, are there $\boldsymbol{c_1, c_2, c_3}$ so that $\boldsymbol{a_0 + a_1 t + a_2 t^2 = c_1(1) + c_2(2+3t) + c_3(4+5t+6t^2)}$?

We would need: $\begin{array}{lrcl} \textbf{Constants} & c_1 + 2c_2 + 4c_3 & = & a_0 \\ t \textbf{ terms} & 3c_2 + 5c_3 & = & a_1 \\ t^2 \textbf{ terms} & 6c_3 & = & a_2 \end{array}$

That is, are there $\boldsymbol{c_1, c_2, c_3}$ so that $c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$**, i.e.** $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$**?**

Yes. (How can you tell there is a solution? Never mind about finding it.) We're reminded that we can think of polynomials (and other non-vector items, as we'll see) as vectors.

Second, are the three polynomials $\{1, 2+3t, 4+5t+6t^2\}$ linearly independent?

We can see from the work above that they are linearly independent, since $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ **are linearly independent. (How can we easily see this?)**

So yes, $\{1, 2+3t, 4+5t+6t^2\}$ is a basis for P_2.

And since we have the right number of polynomials (three), we could have checked either that they span or that they are linearly independent, and the other would automatically have been true.