Most important ideas:

- **Nullspaces and column spaces.**
- **Compare and contrast the two spaces: page 204.**

Describe the nullspace of a matrix A: **The collection of all vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$.**

That is: $\boldsymbol{Nul\ A} = \{\vec{x} : A\vec{x} = \vec{0}\}$

Consider $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{bmatrix}$.

Is $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ in the nullspace of A? **Check:** $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 18 \\ 36 \\ 54 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$**, so $\vec{v}_1$ is <u>not</u> in $Nul\ A$.**

Is $\vec{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ -3 \end{bmatrix}$ in the nullspace of A? **Check:** $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$**, so $\vec{v}_2$ <u>is</u> in $Nul\ A$.**

Is it easy to check whether or not a vector $\vec{v}$ is in $Nul\ A$? **Yes. Just see if $A\vec{v} = \vec{0}$.**

Find another vector in the nullspace of A, or find the entire nullspace of A. **This is more difficult.**

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 0 \\ 4 & 5 & 6 & 1 & 0 \\ 7 & 8 & 9 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ so } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + x_4 \\ -2x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Notice that the vector $\vec{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \\ -3 \end{bmatrix}$ in $Nul\ A$ from above is of this form: $\begin{bmatrix} -1 \\ -1 \\ 2 \\ -3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

Notice that the nullspace of 3×4 matrix A is a subspace of $\boldsymbol{R^4}$ since this is the size of the vectors $\vec{x}$ by which we multiply A to get $\vec{0}$. That is, in $A\vec{x} = \vec{0}$, $\vec{x}$ is from R^4.

In general for a matrix A whose columns are $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$, that is, $A = [\,\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_n\,]$:

Describe the column space of a matrix A: **The set of all vectors that can be built out of the columns of A.**

That is: **$\vec{b} \in Col\ A$ means $\vec{b} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots x_n\vec{a}_n$ for some $x_1, x_2, \dots, x_n$.**

That is: **$\vec{b} \in Col\ A$ means $\vec{b} = A\vec{x}$ for some $\vec{x}$.** That is: **$Col\ A = \{\vec{b} : \vec{b} = A\vec{x}$ for some $\vec{x}\}$.**

Example: Describe/find the column space of $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{bmatrix}$. **$Col\ A$ consists of all vectors of the form $x_1\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} + x_4\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, for some x_1, x_2, x_3, x_4, that is,**

of the form $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, that is, of the form $A\vec{x}$ for some $\vec{x}$.

Is $\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ in $Col\ A$? **That is harder to check. We want to know if there are x_1, x_2, x_3, x_4**

such that $x_1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, **that is, such that** $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

Let's see: $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 1 \\ 4 & 5 & 6 & 1 & 0 \\ 7 & 8 & 9 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$, **which means there is no solution, so** $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ **cannot be built using the columns of A, that is, it is not in $Col\ A$.**

Is $\vec{b}_2 = \begin{bmatrix} -1 \\ 5 \\ 11 \end{bmatrix}$ in the column space of A?

Let's see: $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & -1 \\ 4 & 5 & 6 & 1 & 5 \\ 7 & 8 & 9 & 1 & 11 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 5 \\ 0 & 1 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$, **from which we can see there are actually infinite solutions (infinite ways to build $\vec{b}_2$ using the columns of A:**

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 + x_3 + x_4 \\ -3 - 2x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

(Notice that this is the sum of a particular solution and the homogeneous solutions.)

For example, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \\ 0 \end{bmatrix}$. **Notice that** $\begin{bmatrix} -1 \\ 5 \\ 11 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

It is easy to find vectors that are in the column space: just decide how much of each column of A you want in building your vector. But it is more difficult to check whether $\vec{b}_1$ and $\vec{b}_2$ are in the $Col\ A$. Notice that **the column space of 3×4 matrix A is a subspace of R^3**, since the column space is the collection of vectors that can be built (as linear combinations) of the columns of A, which are from R^3.

	$\vec{b} = A\vec{x}$	for $\vec{x} =$
Some vectors $\vec{b}$ in column space of $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{bmatrix}$:	$\begin{bmatrix} -1 \\ 5 \\ 11 \end{bmatrix}, \begin{bmatrix} 7 \\ 16 \\ 25 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ldots$	$\begin{bmatrix} 5 \\ -3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ldots$

Which is easier to find, a vector in the nullspace or the column space?
See items 2 and 3 on page 204.

Which is easier to check, that a vector is in the nullspace or that it is in the column space?
See items 5 and 6 on page 204.