What happens in vector spaces stays in vector spaces. —Some famous mathematician

Most important ideas:

- **A vector space is a collection of vectors which satisfies the properties on page 190. The most important properties are 1, 4 and 6.**
- **A vector space can consist of things other than vectors, such as polynomials.**

Example 1: Consider the set of points on the plane $x - 2y + 3z = 0$.

Two such points (vectors): $\vec{v}_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}$ or $\vec{v}_2 = \begin{bmatrix}-1\\1\\1\end{bmatrix}$. Try coming up with two of your own.

Notice: that $\vec{v}_1 + \vec{v}_2 = \begin{bmatrix}0\\3\\2\end{bmatrix}$ is also on the plane since $\mathbf{0 - 2(3) + 3(2) = 0.}$

and any multiple of $\vec{v}_1$, such as $\mathbf{7\vec{v}_1 = \begin{bmatrix}7\\14\\7\end{bmatrix}}$, is also on the plane, since $\mathbf{7 - 2(14) + 3(7) = 0.}$

In general, points on the plane are $x - 2y + 3z = 0$:

$$\begin{aligned}\boldsymbol{x} &= \boldsymbol{2y - 3z}\\ \boldsymbol{y} &= \boldsymbol{y}\\ \boldsymbol{z} &= \boldsymbol{z}\end{aligned} \qquad \textbf{that is, } \begin{bmatrix}\boldsymbol{x}\\\boldsymbol{y}\\\boldsymbol{z}\end{bmatrix} = \boldsymbol{y}\begin{bmatrix}\mathbf{2}\\\mathbf{1}\\\mathbf{0}\end{bmatrix} + \boldsymbol{z}\begin{bmatrix}\mathbf{-3}\\\mathbf{0}\\\mathbf{1}\end{bmatrix}.$$

Notice that $\vec{v}_1 = \begin{bmatrix}1\\2\\1\end{bmatrix} = \mathbf{2}\begin{bmatrix}\mathbf{2}\\\mathbf{1}\\\mathbf{0}\end{bmatrix} + \mathbf{1}\begin{bmatrix}\mathbf{-3}\\\mathbf{0}\\\mathbf{1}\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}-1\\1\\1\end{bmatrix} = \mathbf{1}\begin{bmatrix}\mathbf{2}\\\mathbf{1}\\\mathbf{0}\end{bmatrix} + \mathbf{1}\begin{bmatrix}\mathbf{-3}\\\mathbf{0}\\\mathbf{1}\end{bmatrix}$.

Also notice that zero vector is on the plane since $\mathbf{0 - 2(0) + 3(0) = 0.}$

Example 2: Consider the set of points on the plane $x - 2y + 3z = 4$ (or any non-zero right hand side).

Two such points (vectors) are $\vec{v}_1 = \begin{bmatrix}-3\\-2\\1\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}10\\3\\0\end{bmatrix}$. Try finding two of your own vectors.

Notice: that $\vec{v}_1 + \vec{v}_2 = \begin{bmatrix}7\\1\\1\end{bmatrix}$ is <u>not</u> on the plane since $\mathbf{7 - 2(1) + 3(1) \neq 4}$ **.**

and $7\vec{v}_1 = \begin{bmatrix}-21\\-14\\7\end{bmatrix}$ is <u>not</u> on the plane since $\mathbf{-21 - 2(-14) + 3(7) \neq 4}$ **.**

Also notice that the zero vector is <u>not</u> on the plane since $\mathbf{0 - 2(0) + 3(0) \neq 4.}$

A vector space V is a collection of vectors which satisfy the ten properties on page 190, in particular:

Property 1 (closure under addition): $\boldsymbol{\vec{u}, \vec{v} \in V \Rightarrow \vec{u} + \vec{v} \in V}$

Property 4 (inclusion of zero vector): $\boldsymbol{\vec{0} \in V}$

Property 6 (closure under scalar multiplication): $\boldsymbol{\vec{u} \in V \Rightarrow c\vec{u} \in V}$

If H is a subset of vector space V, then H inherits 7 of the 10 properties, but not necessarily Properties 1, 4 and 6.

A subspace H of vector space V is (1) a subset of V and (2) a vector space.

Example 3 (Example 1 again): R^3 is a vector space. All 10 vector space properties are satisfied. The set of all vectors than can be built with $\begin{bmatrix}2\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-3\\0\\1\end{bmatrix}$, i.e., $Span\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\end{bmatrix}\}$, is a subset of R^3, that is, a collection of vectors that come from R^3. But is it a vector space?

The 7 arithmetic properties are automatically inherited, and we saw in Example 1 that the other 3 properties (1, 4 and 6) were also true. So yes, $Span\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\end{bmatrix}\}$ is a subspace of R^3.

It turns out that the span of any set of vectors is a vector space.

See Theorem 1 and its proof on page 194.

A collection of polynomials can be a vector space.

Example 4: The collection of all polynomials is a vector space. Consider the subset $S = \{p(t) : p(t) = a + t^2\}$, that is, the collection of polynomials of the form "some constant plus t^2." For example, $3 + t^2$ and $-4 + t^2$ are both in S. S is a subset of all polynomials simply means that it is a collection of some of the polynomials. But is S a subspace of the collection of all polynomials? Because S is a subset of the polynomials (a vector space), S automatically inherits the 7 arithmetic properties. But what about the other 3 properties?

Property 1. Suppose that you have two polynomials in S: $p_1(t) = a_1 + t^2$ and $p_2(t) = a_2 + t^2$, for some constants a_1 and a_2. The question is whether the sum of $p_1(t)$ and $p_2(t)$ is of the necessary form "some constant plus t^2". Their sum $p_1(t) + p_2(t) = a_1 + a_2 + 2t^2$ is not of the form "some constant plus t^2" since it is not of the form "some constant plus t^2" thus it is not in S. (For example, $3 + t^2 \; + \; -4 + t^2 = -1 + 2t^2$.)

So we already know that S is not a subspace of the polynomials, since to be a subspace, every single property must be satisfied. However, just for practice, we'll check the other two required properties as well.

Property 4. The zero vector of polynomials is $0 + 0t + 0t^2 + 0t^3 + \cdots$. This is not of the form "some constant plus t^2" and thus is not in S.

Property 6. Suppose that you have a polynomial in S: $p(t) = a + t^2$ for some constant a. Then a multiple of it $cp(t) = ca + ct^2$ is not of the form "some constant plus t^2" and thus is not in S (unless the constant $c = 1$, but this property has to be true no matter what the multiple c is).