Most important ideas:

- **Cramer's Rule.**
- **A formula for the inverse of any size matrix: Theorem 8 on page 179.**
- **Determinants as area or volume.**

Notation used in for this section: $A_i(\vec{b})$ is the matrix A but with column i replaced with vector $\vec{b}$.

Cramer's Rule: where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is the solution to $A\vec{x} = \vec{b}$, then

$$x_1 = \frac{\det A_1(\vec{b})}{\det A}, \quad x_2 = \frac{\det A_2(\vec{b})}{\det A}, \quad x_3 = \frac{\det A_3(\vec{b})}{\det A}$$

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$. First, $\det A = 3$. Then the solution, one variable at a time, is:

$$x_1 = \frac{\begin{vmatrix} 1 & 2 & 3 \\ 7 & 5 & 6 \\ 2 & 1 & 0 \end{vmatrix}}{3} = \frac{9}{3} = 3 \qquad x_2 = \frac{\begin{vmatrix} 1 & 1 & 3 \\ 4 & 7 & 6 \\ 1 & 2 & 0 \end{vmatrix}}{3} = \frac{-3}{3} = -1 \qquad x_3 = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 4 & 5 & 7 \\ 1 & 1 & 2 \end{vmatrix}}{3} = \frac{0}{3} = 0$$

Check solution: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$.

We could write

$$\vec{x} = \begin{bmatrix} \dfrac{\begin{vmatrix} 1 & 2 & 3 \\ 7 & 5 & 6 \\ 2 & 1 & 0 \end{vmatrix}}{\det A} \\[2em] \dfrac{\begin{vmatrix} 1 & 1 & 3 \\ 4 & 7 & 6 \\ 1 & 2 & 0 \end{vmatrix}}{\det A} \\[2em] \dfrac{\begin{vmatrix} 1 & 2 & 1 \\ 4 & 5 & 7 \\ 1 & 1 & 2 \end{vmatrix}}{\det A} \end{bmatrix}$$

Recall: Suppose X is the inverse of A. That is, where $X = [\vec{x}_1 \quad \vec{x}_2 \quad \vec{x}_3]$, we have $AX = I$,

that is, $A[\vec{x}_1 \quad \vec{x}_2 \quad \vec{x}_3] = [A\vec{x}_1 \quad A\vec{x}_2 \quad A\vec{x}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

So with $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 0 \end{bmatrix}$, we can find (solve for) the columns of inverse matrix X one column at a time:

$$A\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \qquad A\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \qquad A\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$$

$$\vec{x}_1 = \begin{bmatrix} \dfrac{\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 1 & 0 \end{vmatrix}}{\det A} \\ \dfrac{\begin{vmatrix} 1 & 1 & 3 \\ 4 & 0 & 6 \\ 1 & 0 & 0 \end{vmatrix}}{\det A} \\ \dfrac{\begin{vmatrix} 1 & 2 & 1 \\ 4 & 5 & 0 \\ 1 & 1 & 0 \end{vmatrix}}{\det A} \end{bmatrix} \qquad \vec{x}_2 = \begin{bmatrix} \dfrac{\begin{vmatrix} 0 & 2 & 3 \\ 1 & 5 & 6 \\ 0 & 1 & 0 \end{vmatrix}}{\det A} \\ \dfrac{\begin{vmatrix} 1 & 0 & 3 \\ 4 & 1 & 6 \\ 1 & 0 & 0 \end{vmatrix}}{\det A} \\ \dfrac{\begin{vmatrix} 1 & 2 & 0 \\ 4 & 5 & 1 \\ 1 & 1 & 0 \end{vmatrix}}{\det A} \end{bmatrix} \qquad \vec{x}_3 = \begin{bmatrix} \dfrac{\begin{vmatrix} 0 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 1 & 0 \end{vmatrix}}{\det A} \\ \dfrac{\begin{vmatrix} 1 & 0 & 3 \\ 4 & 0 & 6 \\ 1 & 1 & 0 \end{vmatrix}}{\det A} \\ \dfrac{\begin{vmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 1 & 1 & 1 \end{vmatrix}}{\det A} \end{bmatrix}$$

This leads to the formula for finding the inverse given in Theorem 8 on page 179:

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} \begin{vmatrix} 5 & 6 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \\ -\begin{vmatrix} 4 & 6 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \\ \begin{vmatrix} 4 & 5 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} -6 & -(-3) & -3 \\ -(-6) & -3 & -(-6) \\ -1 & -(-1) & -3 \end{bmatrix}$$

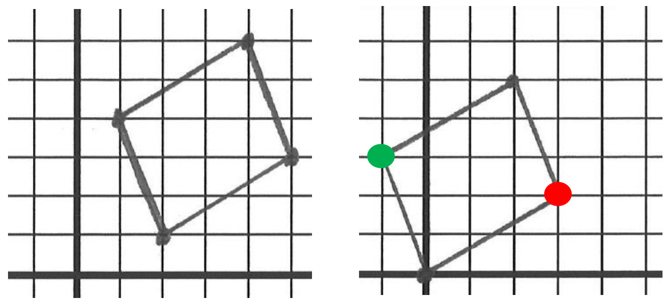
$$= \begin{bmatrix} -2 & 1 & -1 \\ 2 & -1 & 2 \\ -1/3 & 1/3 & -1 \end{bmatrix}$$

If A is the matrix whose columns are the coordinates (relative to the origin) of the vectors which define a parallelogram in R^2 or a parallelepiped in R^3, then $\det A$ is the area or volume of that parallelogram or parallelepiped.

Example: Find the area of the parallelogram whose four vertices are $(2,1), (5,3), (1,4), (4,6)$.

We start by translating (moving) the parallelogram, as shown at right, so that one of its vertices is at the origin and then the two vertices that are connected to the origin are $(3,2)$ and $(-1,3)$ so the area is

$$\begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} = 11.$$



Example: Find the volume of the parallelepiped with one vertex at the origin and whose other three vertices are at $(2,1,1)$, $(3,1,1)$ and $(4,1,-1)$.

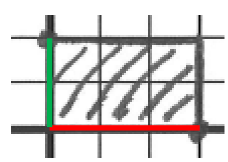
The volume is $\begin{vmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 2$. This is harder to draw, so I won't attempt it. Figure 4 on page 181 gives a drawing of how this sort of thing looks.

This can also be used to determine the "volume" of parallela-things in higher dimensions.

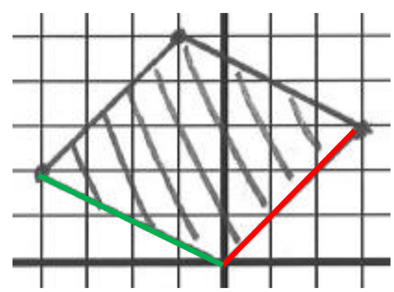
Finally, if S is some shape in R^2 and $T(\vec{x}) = A\vec{x}$ is a linear transformation, then the area of $T(S)$ is $\det A \cdot \text{area of } S$.

Example: Suppose S is the rectangle with vertex at the origin and with width 3 and height 2.

S



$T(S)$



Let $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$.

Then $T\left(\begin{bmatrix} 3 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

And $T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$.

The area of S is 6. The area then of $T(S)$ is $\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} \cdot 6 = 18.$

This is also true for transformations in higher dimensions, even those we can't visualize (in R^3, R^4, etc.).