Most important ideas:

- **How A or B being "bad" is related to AB being "bad."**
- **Various properties listed in Theorems 3 – 6.**
- **The role that elementary matrices play in finding determinants.**
- **You can use Excel or online tools to compute determinants.**

Example 1: Reminder of various properties of determinants.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 0 & -3 & 1 \end{bmatrix},\ AB = \begin{bmatrix} -3 & -5 & 7 \\ -6 & -5 & 16 \\ -1 & 3 & 2 \end{bmatrix},\ A^4 = \begin{bmatrix} 669 & 870 & 945 \\ 1740 & 2265 & 2466 \\ 315 & 411 & 450 \end{bmatrix}$$

$$\det A = 1\begin{vmatrix} 5 & 6 \\ 1 & 0 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 1 & 1 \end{vmatrix} = 3$$

$$\det A^T = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{vmatrix} = 3$$

$$\det A^{-1} = \begin{vmatrix} -2 & 1 & -1 \\ 2 & -1 & 2 \\ -1/3 & 1/3 & -1 \end{vmatrix} = 1/3$$

$\det B = 11$

$\det AB = 33 = \det A \cdot \det B$

$\det A^4 = 81$

Reminders: $\boldsymbol{\det A^T = \det A}$

$$\boldsymbol{\det A^{-1} = \frac{1}{\det A}}$$

$$\boldsymbol{\det AB = \det A \cdot \det B}$$

$$\boldsymbol{\det(A + B) \neq \det A + \det B}$$

$$\boldsymbol{\det\left(A^k\right) = (\det A)^k}$$

Multiply top row (or any row or column) of A by 10:

$$\begin{vmatrix} 10 & 20 & 30 \\ 4 & 5 & 6 \\ 1 & 1 & 0 \end{vmatrix} = \mathbf{10}\begin{vmatrix} \mathbf{5} & \mathbf{6} \\ \mathbf{1} & \mathbf{0} \end{vmatrix} - \mathbf{20}\begin{vmatrix} \mathbf{4} & \mathbf{6} \\ \mathbf{1} & \mathbf{0} \end{vmatrix} + \mathbf{30}\begin{vmatrix} \mathbf{4} & \mathbf{5} \\ \mathbf{1} & \mathbf{1} \end{vmatrix} = \mathbf{30}$$ **(compare this to finding $\det A$ above)**

Multiply entire matrix by 10:

$$\det 10A = \begin{vmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \\ 10 & 10 & 0 \end{vmatrix} = \mathbf{10}\begin{vmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \\ 1 & 1 & 0 \end{vmatrix} = \mathbf{10^2}\begin{vmatrix} 10 & 20 & 30 \\ 4 & 5 & 6 \\ 1 & 1 & 0 \end{vmatrix} = \mathbf{10^3}\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 0 \end{vmatrix} = \mathbf{10^3 \cdot \det A}$$

We can use elementary matrices to help us compute the determinant of a matrix. Reminder:

Operation	Examples	Determinant	Inverses	Determinant
Swap rows	$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$	$-1, \ -1$	$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$	$-1, \ -1$
Multiply a row by k	$\begin{bmatrix}1&0&0\\0&5&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&0\\0&0&-3\end{bmatrix}$	$5, \ -3$	$\begin{bmatrix}1&0&0\\0&1/5&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&0\\0&0&-1/3\end{bmatrix}$	$\frac{1}{5}, \ -\frac{1}{3}$
Add k times one row to another	$\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&0\\0&-3&1\end{bmatrix}$	$1, \ 1$	$\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&0\\0&3&1\end{bmatrix}$	$1, \ 1$

If $E_n \cdots E_2 E_1 A = U$, then $\mathbf{A = (E_n \cdots E_2 E_1)^{-1} U = E_1^{-1} E_2^{-1} \cdots E_n^{-1} U}$
and $\mathbf{\det A = \det E_1^{-1} \cdot \det E_2^{-1} \cdot \ \cdots \ \cdot \det E_n^{-1} \cdot \det U}$
where $\det U$ is the product of the values along the diagonal, assuming U is upper triangular.

(Also recall that $(E_n \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_n^{-1}$ would be the lower triangular matrix L in the LU-factorization $A = LU$.)

Example: Find the determinant of $A = \begin{bmatrix}1&2&3\\4&5&6\\7&8&0\end{bmatrix}$.

Next matrix	Next step	Elementary matrix	Inverse	Det. of inverse
$\begin{bmatrix}1&2&3\\4&5&6\\7&8&0\end{bmatrix}$	$\mathbf{R2 - 4R1}$	$E_1 = \begin{bmatrix}1&0&0\\-4&1&0\\0&0&1\end{bmatrix}$	$E_1^{-1} = \begin{bmatrix}1&0&0\\4&1&0\\0&0&1\end{bmatrix}$	$\det E_1^{-1} = 1$
$\begin{bmatrix}1&2&3\\0&-3&-6\\7&8&0\end{bmatrix}$	$\mathbf{R3 - 7R1}$	$E_2 = \begin{bmatrix}1&0&0\\0&1&0\\-7&0&1\end{bmatrix}$	$E_2^{-1} = \begin{bmatrix}1&0&0\\0&1&0\\7&0&1\end{bmatrix}$	$\det E_2^{-1} = 1$
$\begin{bmatrix}1&2&3\\0&-3&-6\\0&-6&-21\end{bmatrix}$	$\mathbf{R3 - 2R2}$	$E_3 = \begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}$	$E_3^{-1} = \begin{bmatrix}1&0&0\\0&1&0\\0&2&1\end{bmatrix}$	$\det E_3^{-1} = 1$
$\begin{bmatrix}1&2&3\\0&-3&-6\\0&0&-9\end{bmatrix}$	**Done: This is U**			

Then $\det A = \det E_1^{-1} \cdot \det E_2^{-1} \cdot \det E_3^{-1} \cdot \det U = \mathbf{1 \cdot 1 \cdot 1 \cdot (1)(-3)(-9) = 27.}$

$\det A = \det E_1^{-1} \cdot \det E_2^{-1} \cdot \det E_3^{-1} \cdot \det U$ is not an efficient way to compute the determinant of a matrix, but it is a useful theoretical result (but it is faster than the co-factor expansion—see Numerical Note on page 167).

Finally, note that if $L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix}1&0&0\\4&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\7&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&2&1\end{bmatrix} = \begin{bmatrix}1&0&0\\4&1&0\\7&2&1\end{bmatrix}$,

then $A = LU$: $\begin{bmatrix}1&2&3\\4&5&6\\7&8&0\end{bmatrix} = \begin{bmatrix}1&0&0\\4&1&0\\7&2&1\end{bmatrix}\begin{bmatrix}1&2&3\\0&-3&-6\\0&0&-9\end{bmatrix}$. (BTW, why is L lower triangular?)