Math 260 Section 3.1

Most important ideas:

- **The determinant $\det A$ of a square matrix:**
  - $\det A \neq 0$ **means the matrix has an inverse (and all other "good" things are also true).**
  - $\det A = 0$ **means the matrix has no inverse (and all other "bad" things are also true).**
  - $\det A$ **gives us other information about a matrix, which we will learn later.**
- **Definition of how to find the determinant of any size matrix in Theorem 1 on page 166.**
- **A simple formula for the determinant of a 3×3 matrix precedes homework problems 15 – 18.**
- **The determinant of a triangular (lower or upper) matrix is the product of the diagonal entries.**

Recall for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant of A is $ad - bc$, and that $A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
So we see that:

$\det A \neq 0$ means that A is invertible, i.e. has an inverse (and is sometimes called "nonsingular").

$\det A = 0$ means that A is not invertible (and is sometimes called "singular").

This is true in general for any size (square) matrix.

Another notation for determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

For $m \times n$ matrix A, cofactor A_{ij} is the $m - 1 \times n - 1$ submatrix of A without row i and column j.

Example 1: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $A_{23} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$, $A_{31} = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$

How to find the determinant of a larger than 2×2 matrix? Theorem 1 on page 166.

Example 2: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

$$\det A = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + 7\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 1(45 - 48) - 4(18 - 24) + 7(12 - 15) = 0 .$$

Part of this is the pattern of $+$ and $-$, starting with $+$ at the top, left hand position.
For a 3×3 matrix: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$.

You can do this cofactor expansion along any row or column. For example, along column 2:

$$\det A = -2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 0 .$$

Example 3: For $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$, $\det A = 24$.

Observation: the determinant of an upper (or lower) triangular matrix is the product of the values one the diagonal. In the above example, the determinant is simply $\mathbf{1 \cdot 4 \cdot 6 = 24}$.

Example 4: **The determinant of the identity matrix I is 1. For example,** $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot 1 \cdot 1 = 1$ **.**

Example 5: Determinants of elementary matrices

Operation	**Example**	**Determinant**
Swap rows	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	-1
Multiply a row by k	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$	k
Add k times one row to another	$\begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	1

Example 6: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}$, $5A = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}$

$A + B = \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}$, $AB = \begin{bmatrix} 3 & 4 \\ 7 & 10 \end{bmatrix}$,

$\det A = -2$

$\det B = -1$

$\det A^T = -2$

$\det A^{-1} = -1/2$

$\det 5A = -50$

$\det (A+B) = -6$

$\det AB = 2$

$\det A^{-1} = -1/2$

Notice some important properties:

$\det AB = \det A \cdot \det B$

$\det A^{-1} = 1/\det A$

$\det A^T = \det A$

$\det cA = c^n \det A$ (where A is $n \times n$)

$\det (A+B) \neq \det A + \det B$