More on the use of elementary matrices

Given $= \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$, use elementary matrices to find $A = LU$ and A^{-1}.

So let's row reduce A into the identity matrix.

Step	Elementary matrix	Result of step
$R_2 + 3R_1$	$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 2 & -3 & 4 \end{bmatrix}$
$R_3 + (-2)R_1$	$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & -3 & 8 \end{bmatrix}$
$R_3 + 3R_2$	$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$
$1/2\ R_3$	$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$
$R_1 + 2R_3$	$E_5 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$
$R_2 + 2R_3$	$E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

First, we have $E_3\big(E_2(E_1A)\big) = U = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$, i.e. $(E_3E_2E_1)A = U$, so $A = (E_3E_2E_1)^{-1}U$, and

$$L = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}.$$

Check that $LU = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix} = A$. Notice that L is really just the matrix that describes the opposite of what we did in putting A into echelon form.

Second, after all steps we have $E_6\left(E_5\left(E_4\left(E_3\big(E_2(E_1A)\big)\right)\right)\right) = I$, i.e. $(E_6E_5E_4E_3E_2E_1)A = I$, so

$$A^{-1} = E_6E_5E_4E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}.$$

We could also have computed $A^{-1} = (LU)^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 7 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}.$

This is a more efficient way to find A^{-1}, as it is much easier to find inverses of upper and lower triangular matrices.