Math 260 Section 2.6

Most important ideas:

- The Leontief Input-Output Model.
- Increased production of one product results in increased consumption of every product.
- Column i of the consumption matrix C contains the amounts of products consumed if producing one unit of product i.
- Increased demand in one product requires increased production of every product.
- Column i of the inverse of $I-C$ contains the amounts of products to produce if we want to end up with one unit of product i.

Example 1. Suppose that a certain company produces coal and electricity but that some of each product is consumed in the production process:

To produce one unit of		Requires units of
Coal	Electricity	
0	.4	Coal
.2	.1	Electricity

Suppose they produce 10 units of coal and 10 units of electricity. Then consumed in the process is:

$$\begin{array}{ll} \textbf{Coal} & 10(0)+10(.4)=4 \\ \textbf{Electricity} & 10(.2)+10(.1)=3 \end{array}, \text{ i.e., } \begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \end{bmatrix} = 10\begin{bmatrix} 0 \\ .2 \end{bmatrix} + 10\begin{bmatrix} .4 \\ .1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \text{ so remaining is } \begin{bmatrix} 10 \\ 10 \end{bmatrix} - \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}.$$

A few more cases. Compare each case to the first case of producing $\begin{bmatrix} 10 \\ 10 \end{bmatrix}$.

Produced	Consumed	Observation (compare each case to the original amount consumed $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$)
$\begin{bmatrix} 10 \\ 10 \end{bmatrix}$	$\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$	
$\begin{bmatrix} 20 \\ 20 \end{bmatrix}$	$\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$	**Doubling amount produced of all items results in doubling total amount consumed of all items.**
$\begin{bmatrix} 10 \\ 20 \end{bmatrix}$	$\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$	**Doubling amount produced of <u>one</u> item results in more consumption of <u>all</u> items (but not necessarily doubling of any item).**
$\begin{bmatrix} 20 \\ 10 \end{bmatrix}$	$\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$	**Doubling amount produced of <u>one</u> item results in more consumption of <u>all</u> items (but not necessarily doubling of any item).**
$\begin{bmatrix} 60 \\ 10 \end{bmatrix}$	$\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 60 \\ 10 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \end{bmatrix}$	**This case isn't possible—we would use more of item two (electricity) than we produce. We need to produce more of item two.**
$\begin{bmatrix} 11 \\ 10 \end{bmatrix}$	$\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 11 \\ 10 \end{bmatrix} = \begin{bmatrix} 4 \\ 3.2 \end{bmatrix}$	**Producing 1 more unit of item one (coal) consumes $\begin{bmatrix} 0 \\ .2 \end{bmatrix}$ more of $\begin{bmatrix} coal \\ elect. \end{bmatrix}$. Compare this to the next case. This is column one of the consump. matrix.**
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ .2 \end{bmatrix}$	**Producing $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, i.e. 1 unit of item one, consumes $\begin{bmatrix} 0 \\ .2 \end{bmatrix}$, which is simply column one of the consumption matrix. Compare this to the previous case.**
$\begin{bmatrix} 10 \\ 11 \end{bmatrix}$	$\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 10 \\ 11 \end{bmatrix} = \begin{bmatrix} 4.4 \\ 3.1 \end{bmatrix}$	**Producing 1 more unit of item two (elect.) consumes $\begin{bmatrix} .4 \\ .1 \end{bmatrix}$ more of $\begin{bmatrix} coal \\ elect. \end{bmatrix}$. Compare this to the next case. This is column two of consump. matrix.**
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .4 \\ .1 \end{bmatrix}$	**Producing $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, i.e. 1 unit of item two, consumes $\begin{bmatrix} .4 \\ .1 \end{bmatrix}$, which is simply column two of the consumption matrix. Compare this to the previous case.**

Observation: In general, column j of the consumption matrix is the amounts consumed if we produce 1 unit of item j and none of any of the other items.

Problem 1: How much to produce if we want to end up with 10 units of each? We've seen that producing 10 units of each is obviously not enough. We've also seen that producing 20 units of each is too much.

Producing $\begin{bmatrix} 10 \\ 10 \end{bmatrix}$ **consumes** $\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ **so there is** $\begin{bmatrix} 10 \\ 10 \end{bmatrix} - \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$ **remaining.**

Producing $\begin{bmatrix} 20 \\ 20 \end{bmatrix}$ **consumes** $\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ **so there is** $\begin{bmatrix} 20 \\ 20 \end{bmatrix} - \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 12 \\ 14 \end{bmatrix}$ **remaining.**

Since we want to have $\begin{bmatrix} 10 \\ 10 \end{bmatrix}$ **remaining, we expect to produce between 10 and 20 units of each.**

So how much? In general, where: $\vec{x}$ is the amount produced

C is the consumption matrix

So that: $C\vec{x}$ is the amount consumed

And where: $\vec{d}$ is the demand (the amount we want to end up with)

Then we want/need: Produced – Consumed = Demand

That is: $\boldsymbol{I\vec{x} - C\vec{x} = \vec{d}}$

We can solve for $\vec{x}$: $\boldsymbol{(I - C)\vec{x} = \vec{d} \Rightarrow \vec{x} = (I - C)^{-1}\vec{d}}$

Problem 1 continued: So how much to produce if we want to end up with demand $\vec{d} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$?

First: $\boldsymbol{I - C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix} = \begin{bmatrix} 1 & -.4 \\ -.2 & .9 \end{bmatrix}$

Next: $\begin{bmatrix} 1 & -.4 \\ -.2 & .9 \end{bmatrix}^{-1} = \frac{1}{(1)(.9)-(-.2)(-.4)}\begin{bmatrix} .9 & .4 \\ .2 & 1 \end{bmatrix} = \frac{1}{.82}\begin{bmatrix} .9 & .4 \\ .2 & 1 \end{bmatrix} \approx \begin{bmatrix} 1.098 & 0.488 \\ 0.244 & 1.220 \end{bmatrix}$

Finally: $\boldsymbol{\vec{x} = (I - C)^{-1}\vec{d}} = \begin{bmatrix} 1.098 & 0.488 \\ 0.244 & 1.220 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 15.854 \\ 14.634 \end{bmatrix}$

A few more cases. Compare each case to the first case, when demand is $\begin{bmatrix} 10 \\ 10 \end{bmatrix}$.

Demand	Amount to produce	Consumed	Remaining
$\begin{bmatrix} 10 \\ 10 \end{bmatrix}$	$\begin{bmatrix} 1.098 & .488 \\ .244 & 1.220 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 15.854 \\ 14.634 \end{bmatrix}$	$\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 15.854 \\ 14.634 \end{bmatrix} = \begin{bmatrix} 5.854 \\ 4.634 \end{bmatrix}$	$\begin{bmatrix} 10 \\ 10 \end{bmatrix}$
$\begin{bmatrix} 20 \\ 20 \end{bmatrix}$	$\begin{bmatrix} 1.098 & .488 \\ .244 & 1.220 \end{bmatrix}\begin{bmatrix} 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 31.707 \\ 29.268 \end{bmatrix}$	$\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 31.707 \\ 29.268 \end{bmatrix} = \begin{bmatrix} 11.707 \\ 9.268 \end{bmatrix}$	$\begin{bmatrix} 20 \\ 20 \end{bmatrix}$
$\begin{bmatrix} 20 \\ 10 \end{bmatrix}$	$\begin{bmatrix} 1.098 & .488 \\ .244 & 1.220 \end{bmatrix}\begin{bmatrix} 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 26.829 \\ 17.073 \end{bmatrix}$	$\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 26.829 \\ 17.073 \end{bmatrix} = \begin{bmatrix} 6.829 \\ 7.073 \end{bmatrix}$	$\begin{bmatrix} 20 \\ 10 \end{bmatrix}$
$\begin{bmatrix} 10 \\ 20 \end{bmatrix}$	$\begin{bmatrix} 1.098 & .488 \\ .244 & 1.220 \end{bmatrix}\begin{bmatrix} 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 20.732 \\ 26.829 \end{bmatrix}$	$\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 20.732 \\ 26.829 \end{bmatrix} = \begin{bmatrix} 10.732 \\ 6.829 \end{bmatrix}$	$\begin{bmatrix} 10 \\ 20 \end{bmatrix}$
$\begin{bmatrix} 60 \\ 10 \end{bmatrix}$	$\begin{bmatrix} 1.098 & .488 \\ .244 & 1.220 \end{bmatrix}\begin{bmatrix} 60 \\ 10 \end{bmatrix} = \begin{bmatrix} 70.732 \\ 26.829 \end{bmatrix}$	$\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 70.732 \\ 26.829 \end{bmatrix} = \begin{bmatrix} 10.732 \\ 6.829 \end{bmatrix}$	$\begin{bmatrix} 60 \\ 10 \end{bmatrix}$
$\begin{bmatrix} 11 \\ 10 \end{bmatrix}$	$\begin{bmatrix} 1.098 & .488 \\ .244 & 1.220 \end{bmatrix}\begin{bmatrix} 11 \\ 10 \end{bmatrix} = \begin{bmatrix} 16.951 \\ 14.878 \end{bmatrix}$	$\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 16.951 \\ 14.878 \end{bmatrix} = \begin{bmatrix} 5.951 \\ 4.878 \end{bmatrix}$	$\begin{bmatrix} 11 \\ 10 \end{bmatrix}$
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1.098 & .488 \\ .244 & 1.220 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.098 \\ 0.244 \end{bmatrix}$	$\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 1.098 \\ 0.244 \end{bmatrix} = \begin{bmatrix} 0.098 \\ 0.244 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 10 \\ 11 \end{bmatrix}$	$\begin{bmatrix} 1.098 & .488 \\ .244 & 1.220 \end{bmatrix}\begin{bmatrix} 10 \\ 11 \end{bmatrix} = \begin{bmatrix} 16.341 \\ 15.854 \end{bmatrix}$	$\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 16.341 \\ 15.854 \end{bmatrix} = \begin{bmatrix} 6.341 \\ 4.854 \end{bmatrix}$	$\begin{bmatrix} 10 \\ 11 \end{bmatrix}$
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1.098 & .488 \\ .244 & 1.220 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.488 \\ 1.220 \end{bmatrix}$	$\begin{bmatrix} 0 & .4 \\ .2 & .1 \end{bmatrix}\begin{bmatrix} 0.488 \\ 1.220 \end{bmatrix} = \begin{bmatrix} 0.488 \\ 0.220 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Observation: In general, column j of the matrix $(I - C)^{-1}$ is the amounts we need to produce if we want to end up with 1 unit of item j and none of any of the other items.

A result that we'll use on the next page: $I + C + C^2 + C^3 + \cdots = (I - C)^{-1}$

Proof that $1 + a + a^2 + \cdots a^n = \frac{1-a^{n+1}}{1-a}$.

Let $S_n = 1 + a + a^2 + \cdots + a^n$.

Then $1 + aS_n = 1 + a + a^2 + \cdots + a^n + a^{n+1}$

So that $aS_n = a + a^2 + \cdots + a^n + a^{n+1}$

Then $S_n(1-a) = 1 - a^{n+1}$ which means $S_n = \frac{1-a^{n+1}}{1-a}$.

Useful corollary: Suppose $|a| < 1$, so that $a^{n+1} \to 0$ as $n \to \infty$. For $S_n = \frac{1-a^{n+1}}{1-a}$ letting $n \to \infty$ gives

$$\mathbf{1 + a + a^2 + \cdots = \frac{1-0}{1-a} = (1-a)^{-1}}.$$

Two examples: $a = \frac{1}{2}$ $\quad \mathbf{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2}$

$a = -\frac{1}{3}$ $\quad \mathbf{1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots = \frac{1}{1-(-\frac{1}{3})} = \frac{1}{\frac{4}{3}} = \frac{3}{4}}$

There is a matrix version of this. If C is a matrix such that $C^n \to 0$ (the zero matrix) as $n \to \infty$ then

$$\boldsymbol{I + C + C^2 + \cdots = (I - C)^{-1}}$$

Example:

$$C = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1/2 \end{bmatrix}$$

$$C^2 = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/4 & 2/4 \\ 0 & 1/4 \end{bmatrix}$$

$$C^3 = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1/4 & 2/4 \\ 0 & 1/4 \end{bmatrix} = \begin{bmatrix} 1/8 & 3/8 \\ 0 & 1/8 \end{bmatrix}$$

Etc.

Then

$$I + C + C^2 + C^3 + \cdots = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} + \begin{bmatrix} \mathbf{1/2} & \mathbf{1/2} \\ \mathbf{0} & \mathbf{1/2} \end{bmatrix} + \begin{bmatrix} \mathbf{1/4} & \mathbf{2/4} \\ \mathbf{0} & \mathbf{1/4} \end{bmatrix} + \begin{bmatrix} \mathbf{1/8} & \mathbf{3/8} \\ \mathbf{0} & \mathbf{1/8} \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} \mathbf{1 + 1/2 + 1/4 + 1/8 + \cdots} & \mathbf{0 + 1/2 + 2/4 + 3/8 + \cdots} \\ \mathbf{0} & \mathbf{1 + 1/2 + 1/4 + 1/8 + \cdots} \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} \end{bmatrix}$$

Compare to:

$(I - C)^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 0 & 1/2 \end{bmatrix}$, so $(I - C)^{-1} = \frac{1}{(1/2)(1/2)-(0)(-1/2)}\begin{bmatrix} 1/2 & 1/2 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$.

(Fun but not critical question: how can we show that $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots = 2$?)

So we've shown that (under certain conditions) $I + C + C^2 + C^3 + \cdots = (I - C)^{-1}$.

Another approach to this input-output consumption problem:

Day 1: Produce the desired amount, and some will be consumed

Day 2: Produce the amount we're still short (the amount that was consumed on day 1)

Day 3: Produce the amount that we're still short (the amount that was consumed on day 2)

And so on.

Suppose we want to end up with 10 units of each. So on Day 1 we produce 10 units of each. We consume some, so we don't have enough. On Day 2 we produce this missing amount. And so on…

Day	Produced	Consumed	Total produced so far
1	$\begin{bmatrix}10\\10\end{bmatrix}$	$\begin{bmatrix}0 & .4\\.2 & .1\end{bmatrix}\begin{bmatrix}10\\10\end{bmatrix} = \begin{bmatrix}\mathbf{4}\\\mathbf{3}\end{bmatrix}$	$\begin{bmatrix}\mathbf{10}\\\mathbf{10}\end{bmatrix}$
2	$\begin{bmatrix}\mathbf{4}\\\mathbf{3}\end{bmatrix}$	$\begin{bmatrix}0 & .4\\.2 & .1\end{bmatrix}\begin{bmatrix}\mathbf{4}\\\mathbf{3}\end{bmatrix} = \begin{bmatrix}\mathbf{1.2}\\\mathbf{1.1}\end{bmatrix}$	$\begin{bmatrix}\mathbf{14}\\\mathbf{13}\end{bmatrix}$
3	$\begin{bmatrix}\mathbf{1.2}\\\mathbf{1.1}\end{bmatrix}$	$\begin{bmatrix}0 & .4\\.2 & .1\end{bmatrix}\begin{bmatrix}\mathbf{1.2}\\\mathbf{1.1}\end{bmatrix} = \begin{bmatrix}\mathbf{.44}\\\mathbf{.35}\end{bmatrix}$	$\begin{bmatrix}\mathbf{15.2}\\\mathbf{14.1}\end{bmatrix}$
4	$\begin{bmatrix}\mathbf{.44}\\\mathbf{.35}\end{bmatrix}$	$\begin{bmatrix}0 & .4\\.2 & .1\end{bmatrix}\begin{bmatrix}\mathbf{.44}\\\mathbf{.35}\end{bmatrix} = \begin{bmatrix}\mathbf{.140}\\\mathbf{.35}\end{bmatrix}$	$\begin{bmatrix}\mathbf{15.64}\\\mathbf{14.45}\end{bmatrix}$

In general, if we want to end up with $\vec{d}$. Then produce $\vec{d}$ on day 1, and so on…

Day	Produced	Consumed	Total produced so far
1	$\vec{d}$	$C\vec{d}$	$\vec{d}$
2	$C\vec{d}$	$C(C\vec{d}) = C^2\vec{d}$	$\vec{d} + C\vec{d}$
3	$C^2\vec{d}$	$C^3\vec{d}$	$\vec{d} + C\vec{d} + C^2\vec{d}$
4	$C^3\vec{d}$	$C^4\vec{d}$	$\vec{d} + C\vec{d} + C^2\vec{d} + C^3\vec{d}$

So we see that the total produced (after an infinite number of days) would be:

$$\vec{d} + C\vec{d} + C^2\vec{d} + C^3\vec{d} + \cdots = (I + C + C^2 + C^3 + \cdots)\vec{d} = (I - C)^{-1}\vec{d}$$

BTW, the identity $I + C + C^2 + \cdots = (I - C)^{-1}$ is useful outside of this particular problem.

This Leontief Input-Output model is more interesting and important (1973 Nobel Prize in Economics—four of his students also won the Nobel Prize) when dealing with more products/sectors, such as how the US economy and the Chinese economy (and all of the other world economies) interact.